Three essays in financial economics

Abstract

PART 1: Stock price with consumption CAPM: an international comparison

PART 2: Time variations in the equity premium: is it habit formation or loss aversion?

PART 3: Common exposure and systemic risk in the banking sector

Three Essays in Financial Economics

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The Faculty of Economics, Business Administration and Information Technology of the University of Zurich hereby authorises the printing of this Doctoral Thesis, without thereby giving any opinion on the views contained therein.

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The Dean: Prof. Dr. H. P. Wehrli

Preface

This thesis is composed of three distinct essays with relatively disparate topics. Each essay constitutes an entity on its own and can therefore be read independently from the others. The first essay proposes a new method to compute the fundamental price of stocks. It also studies in detail the empirical link between the price observed on international markets and the estimated fundamental price. The second essay assesses the ability of recent behavioural models to match the path of excess returns observed on the American stock markets. Finally, the third essay, written with Nicole Allenspach, examines the evolutions of banks' common exposure to shocks and of systemic risk in the banking sector since 1993, as well as the link between them.

During the long journey through this thesis, I constantly benefited from the support of my advisor Thorsten Hens. I am particularly thankful to him for systematically including me in the activities of his academic group, even if my particular situation as external PhD student tended to keep me away from university. I also thank him for giving me enough time to explore several ideas before fixing my attention on the three topics presented in this thesis. I am also extremely grateful to my co-advisor Mathias Hoffmann for his very pertinent comments, especially on the first essay of this thesis.

I was also privileged to benefit from the very helpful and supportive environment of the Financial Stability Unit at the Swiss National Bank. I am particularly grateful to Bertrand Rime, who allowed me time and freedom to work on this thesis, probably sometimes at the expense of more "systemic-relevant" projects. I also thank Robert Bichsel for its relaxed, but dedicated, supervision during these years. This experience would not have been as interesting as it was without the motivating and stimulating conversations with my colleagues Ulrike Bilgram, Urs Birchler, Jürg Blum, Dorothe Bonjour, Matteo Facchinetti, Sonja Gerber, Jeannette Henggeler-Müller, Christian Hott, Hansjörg Lehmann, and Reto Nyffeler. I particularly enjoyed the collaboration with Nicole Allenspach, who worked with me on the third essay of this thesis.

I also would like to thank my parents, Jacqueline and Daniel Monnin, to whom I am deeply indebted, not only for their support during these last years. They have always been a leading example for me and this thesis is, in large part, the result of the exceptional and stimulating intellectual environment that they provided me with during my life.

Finally, I would probably never have completed this work without my wife Nellie by my side during the last seven years. More than anybody else, she experienced with me the highs and lows of writing this thesis. Her constant encouragements and infinite patience were a great support and mean more than I can say.

Zurich, March 2008 Pierre Monnin

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Part I

Stock Price with Consumption CAPM: An International Comparison ⋆

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Abstract

What is the fundamental value of a stock and do prices deviate from it? This paper answers these questions by using a Consumption-Capital Asset Pricing Model. I first show how to express the fundamental price as a function of expected future dividends and consumption as well as of their future conditional variance and covariance. Secondly, I estimate the fundamental price for the United States, the United Kingdom, Japan and Switzerland from 1965 through to 2006. I focus in particular on the impact of the decreasing inflation observed in this period, through structural changes in dividends and in risk premiums or through money illusion. Thirdly, I show that the gap between the price and its fundamental value decreases after a shock, which suggests a link between them. Finally, I show that forecasts using the fundamental price are more accurate than forecasts based on the observed price only or based on simpler fundamental models.

Key words: Fundamental stock price, Consumption CAPM, Inflation, Out-of-sample forecasts.

JEL classification: D53, E44, G12.

⋆ The opinions expressed herein are those of the author and do not necessarily reflect the views of the Swiss National Bank.

1 Introduction

What is a stock really worth? Do stock prices deviate from this fundamental (or fair) value? And if they deviate, do they eventually go back to their fundamental value? Such questions have been prominent topics for decades in the finance profession. Many economic agents are interested in their answers: shareholders, who are comparing investment alternatives, traders, who are looking for speculation opportunities, or central bankers, who try to identify stock market imbalances that could threaten the economy. In this paper, I study these three questions with a stock valuation model based on the Consumption-Capital Asset Pricing Model (C-CAPM) (Lucas, 1978, Breeden, 1979). I first show how to use the no-arbitrage condition of the C-CAPM to express the fundamental stock price as a function of expected dividends and consumption, as well as of their covariance. I then estimate the fundamental stock price (and deviations from it) for four countries: the United States, the United Kingdom, Japan and Switzerland. I focus in particular on the impact of the decrease in inflation observed in the eighties on stock prices. I estimate this impact via the changes in real dividend growth and in risk premium that decreasing inflation can cause and via the possible money illusion that it can create. Finally, I assess the out-of-sample forecasting accuracy of the C-CAPM fundamental model. I find that forecasts based on the C-CAPM fundamental price significantly outperform forecasts based on the observed price or on indicators that are known to have some predictive power (e.g. the price-to-dividend ratio).

Not surprisingly given the interest that fundamental stock prices arouse, academics or practitioners have proposed several models to estimate them. A large majority of them are based on the discounted cash flow model (or net present value model), which states that the fundamental stock price is equal to the sum of the discounted expected payoffs of the stock. [1] This type of model requires a forecast of future cash flows generated by the stock, along with an appropriate discount rate. The most basic model in this category is the Gordon growth model (Gordon and Shapiro, 1956, Gordon, 1962). In this model, stocks' payoffs are the dividends. Future discount rates and the future dividend growth rate are both constant. Many authors have refined this model. [2] The first line of innovation is to use dividend forecasts that are more realistic than a constant growth rate. For example, Shiller (1981, 2005) uses ex-post realised dividends.

[1] Lee (1998), Dupuis and Tessier (2003), Zhong, Darrat, and Anderson (2003) and Borio and Lowe (2002) are some of the few exceptions to the net present value model (although the former three indirectly build their method on it). They all measure the fundamental price by separating the permanent component of stock prices from their temporary and non-fundamental component.

[2] An exhaustive survey of the literature on fundamental prices is beyond the scope of this paper. Only a selective list of the main innovations is presented here.

Kaplan and Ruback (1995), Becchetti and Mattesini (2005) or Bagella, Becchetti, and Adriani (2005) use a two-stage model, in which short-term forecasts are given by analysts and long-term forecasts are determined by the historical growth rate. A similar three-stage model is proposed by Panigirtzoglou and Scammell (2002). Yao (1997) separates increasing dividends from decreasing dividends. The second line of developments concentrates on the definition of future cash flow. Ang and Liu (2001), Vuolteenaho (2002) or Dong and Hirshleifer (2005) use earnings instead of dividends; Black, Fraser, and Groenewold (2003a,b) use profits. Cohen, Polk, and Vuolteenaho (2003) or Pástor and Veronesi (2006) choose the market-to-book value ratio instead of the price-dividend ratio (PD ratios hereafter). The third line of improvement concerns the econometric methodology with the use of panel studies rather than single time series (Lee, Myers, and Swaminathan, 1999, Becchetti and Adriani, 2004, Gentry, Jones, and Mayer, 2004). Lastly, some authors have studied the net present value model in a general equilibrium framework (Black, Fraser, and Groenewold, 2003a,b, Kinley, 2004). In contrast to the dividends dynamic, the dynamic of the discount factor has received little attention. In general, the discount factor is constant and is estimated by the CAPM. Campbell and Shiller (1987, 1988b,a) have filled this gap by modelling the dynamic of both dividends and interest rates with a VAR model. They use the estimated joint dynamic to obtain a proxy of agents' expectations. Their VAR approach is the starting point of an impressive body of literature (not all of which is devoted to the estimation of fundamental values).

The fundamental model presented in this paper is in the spirit of the VAR fundamental model developed by Campbell and Shiller (CS hereafter). Like the models cited previously, it is based on the discounted cash-flow model, but it differs from them by using a stochastic discount factor (SDF) based on the no-arbitrage condition of the C-CAPM (Lucas, 1978, Breeden, 1979). In most of the papers cited above, the SDF is given by the traditional CAPM. I use the C-CAPM for two main reasons: firstly, this model is based on sound economic arguments explaining consumption and investment decisions of a representative agent in a general model of a production economy (see Breeden, 1979, Cox, Ingersoll, and Ross, 1985). Secondly, the model links consumption to asset prices. It is therefore well adapted to studying the relationship between the real economy and financial markets. These characteristics have made the C-CAPM one of the cornerstones of asset pricing (see e.g. Cochrane, 2001). To summarize, in the C-CAPM, a rational representative agent splits her income between consumption and savings in a risky asset in order to maximize the utility of both her present and expected future consumption. The no-arbitrage equation states that the utility lost in investing one unit of consumption in an asset today must be equal to the expected utility of the additional future consumption obtained with the asset's payoff. With this condition, the asset price is equal to the expected future payoffs discounted with the intertemporal marginal rate of substitution of the representative agent. This rate is a function of the marginal utility of present and future consumption and thus, indirectly, it is a function of present and future consumption.[3] Consequently, the fundamental asset price is a function of the expected present value of future dividends and future consumption. The fundamentals variables (or fundamentals) are thus dividends and consumption instead of dividends and interest rates as in CAPM-based models. To my knowledge, only Campbell and Shiller (1988b) and Lund and Engsted (1996, LE hereafter) have used the C-CAPM to compute a fundamental stock price.[4]

The model presented here differs from CS and LE in several ways: firstly, I compute three alternative fundamental models, one with stable inflation, another in which inflation is characterized by *structural breaks* and a last one in which agents are subject to *money illusion*. The latter two models are used to study the effects of the significant reduction in inflation observed in the eighties on fundamental stock prices. Both models are a plausible explanation for the general increase in PD ratios observed in the four countries studied in this paper. In the first one, low inflation is associated with low interest rates and less uncertainty in the economy, which should induce higher stock prices. In the second one, money illusion prompts agents to use nominal variables instead of real ones, and thus a decrease in inflation generates a decrease in nominal discount rates, which increases stock prices. Several authors have documented the negative correlation between inflation and stock prices. However, the channel by which inflation influences stock prices is still subject to debate. Recently, several authors have attributed it to money illusion (Ritter and Warr, 2002, Campbell and Vuolteenaho, 2004, Cohen, Polk, and Vuolteenaho, 2005). I find that money illusion seems to play a significant role in the United States and, to a lesser extent, in the United Kingdom, whereas it is less relevant for Japan and Switzerland.

Secondly, the fundamental price developed in this paper is based on a second-order Taylor approximation of the no-arbitrage condition. CS and LE stop at the first-order approximation. With an additional order of approximation the fundamental price becomes a function of the first and second moments of dividends and consumption, including their covariance. I estimate these second moments with a multivariate GARCH model. The advantage of this approach is to capture the impact of *time-varying covariances* on the fundamental price. This is new in the context of fundamental stock prices. Note that the second-order approximation presented in this paper is not restricted to the C-CAPM

[3] The intertemporal marginal rate of substitution also depends on the form of the utility function and thus on the risk aversion of the representative agent.

[4] Shiller (2005) also presents a fundamental price based on the C-CAPM. However, his computation is based on ex-post dividends and consumption growth rate and the coefficient of relative risk aversion is not estimated but arbitrarily set equal to 3.

framework and can be used to compute the fundamental price derived from any other SDF model.

Thirdly, in each step of the computation of the fundamental price for time t, I have been particularly careful to use only the information available at that time. Thus, I put myself in the same position as an investor, who computes the fundamental price at time t. As a result, the fundamental price estimated here is a true *ex-ante* price. CS and LE compute an *ex-post* price by using the whole sample to estimate the fundamentals' dynamic.

Finally, I assess the out-of-sample accuracy of forecasts based on the fundamental price and compare it with other simpler fundamental models. The ability of simple fundamental models to forecast future prices in the long term is now well documented. Campbell and Shiller (1998, 2001) or Rapach and Wohar (2005), for example, show that price-dividends (PD) ratios can help to forecast stock price movements (in-sample) for horizons of 6 to 10 years. Recently, Rapach and Wohar (2006) bring evidence that this result also holds out-of-sample. One of the main results of this paper is to show that the C-CAPM fundamental price is able to *improve out-of-sample forecasts even for horizons shorter than 6 years.* The improvement is particularly spectacular for the United Kingdom, where the fundamental model performs at least 20% better[5] than the random walk with drift for all horizons longer than one year! No other simpler fundamental model tested in this paper is able to systematically outperform the forecasts of the C-CAPM model. In addition, I show that, in the United States, in the United Kingdom and in Switzerland, the accuracy of the C-CAPM fundamental model increases when the price is far from its empirical value. This suggests that the tendency of the price to move back toward its fundamental value is stronger when the misalignment is large. The fact that the C-CAPM fundamental price is able to give out-of-sample forecasts is a sign that there is a link between market price and the fundamental price.

The paper is structured as follow: Section 2 presents the fundamental price equation and the transformations that are necessary to estimate it. Section 3 describes the data. Section 4 documents the long term evolution of PD ratios and introduces the different inflation models used in this paper. Section 5 describes the econometric methodology used to estimate the different coefficients of the fundamental equation. Section 6 presents the estimated fundamental prices. Section 7 checks whether prices eventually go back to their fundamental value and assesses the forecasting performance of the fundamental model. Section 8 assesses the value added of the C-CAPM fundamental model developed in this paper. Section 9 concludes.

[5] Accuracy is measured in terms of mean absolute error.

2 The fundamental stock price equation

The fundamental stock price is based on the no-arbitrage equation derived from the C-CAPM (Section 2.1). From this equation, I express the fundamental price as the present value of expected future fundamentals, which are the dividends and the expected marginal utility of consumption (Section 2.2). Section 2.3 shows how to linearize the fundamental equation and to express the fundamental price as a linear function of expected fundamentals. Section 2.4 explains how to compute the expectations regarding fundamentals. Finally, Section 2.5 combines these elements to give the final equation for the fundamental price.

2.1 The C-CAPM no-arbitrage equation

The first step for computing the fundamental price of stocks is to choose the fundamental model for the market. In this paper, I define the fundamental price as follows

Definition 1 *At time t, the fundamental price of an asset is the equilibrium price resulting from the optimal choice made by a rational representative agent who allocates her income between consumption and savings (in the asset) in order to maximize the utility of her present and expected future real consumption.*

This definition corresponds to the maximization problem at the center of the C-CAPM (Lucas, 1978, Breeden, 1979). The Euler equation given by the first order condition of this maximization problem is

$$P_t = E_t \left[M_{t+1} \left(P_{t+1} + D_{t+1} \right) \right] \tag{1}$$

with

$$M_{t+1} = \beta \frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)} \tag{2}$$

where P_t is the stock price at time t, D_{t+1} is the dividend paid by the stock at the end of period t, β is the subjective discount factor of the representative agent and $U'\left(C_t\right)$ is the marginal utility of consumption C_t in period t. M_{t+1} is called the stochastic discount factor (SDF). This equation is a no-arbitrage equation which states that the utility lost by reducing consumption of one unit in period t and investing it in the stock is equal to the discounted and expected increase in utility obtained from the extra payoff at time $t+1$.

To be able to compute SDF, I assume the following:

Assumption 1 *The representative agent has a power utility function*

With a power utility function, the SDF is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \tag{3}$$

where γ is the coefficient of relative risk aversion of the representative agent.

Of course, many other utility functions are possible (e.g. exponential utility, habit formation, prospect utility, etc...). In particular, utility functions with time-varying risk aversion have attracted a lot of attention in the recent literature.[6] Their success is mainly due to their ability to capture cyclical variation in the SDF. However, the goal of this paper is to estimate the long-term fundamental value of stock prices rather than explaining their short-term variations. In that sense, the constant relative risk aversion implied by power utility is "inherently attractive and is required to explain the stability of financial variables in the face of secular economic growth" (Campbell and Viceira, 2002, p.25). However, if there is a need to use another utility function, the methodology proposed here can easily be adapted by replacing the SDF with the appropriate expression.[7] The only requirement is that the utility function should give a log SDF which is a linear function of observable variables. Habit formation functions or loss aversion functions, for example, have an SDF that fits into this framework (cf. Monnin, 2008).

2.2 Fundamental present value equation

By forward iteration of the future price in the no-arbitrage equation (1), the fundamental price can be expressed as[8]

$$P_t = E_t \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} M_{t+j} \right) D_{t+i} \tag{4}$$

This equation simply tells us that the fundamental asset price in period t is equal to the *expected present value of future dividends* paid by the asset. The discount factor M_{t+i} used to compute the present value is a function of the

[6] For example, habit formation models initiated by Constantinides (1990) or Campbell and Cochrane (1999) have been used extensively to empirically study time-varying risk aversion.

[7] Most asset pricing models can be expressed in the SDF model. They differ only in terms of the form taken by M_{t+1} (cf. Cochrane, 2001).

[8] Formally, the transversality condition $\lim_{i\to\infty} E_t \left[\left(\prod_{j=1}^{i} M_{t+j} \right) P_{t+i} \right] = 0$ is imposed to obtain equation (4). This condition rules out bubbles in the infinite horizon.

expected marginal utilities of future consumption. Thus, the two fundamental variables driving the fundamental asset price are dividends and consumption.

Since prices and dividends are not stationary, it is convenient, for empirical purposes, to express the present value in equation (4) in terms of stationary variables. For that, as suggested by Cochrane (1992), we divide both sides by D_t to get

$$PD_t = E_t \sum_{i=1}^{\infty} \prod_{j=1}^{i} M_{t+j}\gamma_{t+j} \tag{5}$$

where $PD_t = P_t/D_t$ is the price dividend ratio (PD ratio) at time t and $\gamma_t = D_t/D_{t-1}$ is the gross growth rate of dividends between t and $t-1$.

2.3 Linearization

The right-hand side of equation (5) is clearly non-linear, which is not convenient for empirical estimations. To overcome with this problem, it is possible to linearize the fundamental price by taking the logarithm of equation (5) and by using a second-order Taylor expansion of the right-hand side of this equation around its mean. This yields (see proof in Appendix A.1)

$$pd_t = E_t\left(pd_t^*\right) + \frac{1}{2}V_t\left(pd_t^*\right) + R_t \tag{6}$$

where $pd_t = \ln PD_t$ and $pd_t^* = \ln PD_t^*$ with $PD_t^* = \sum_{i=1}^{\infty}\prod_{j=1}^{i} M_{t+j}\gamma_{t+j}$ being the PD ratio with all future fundamentals known with certainty (i.e. the PD ratio with perfect forecasts). R_t is the remainder of the Taylor expansion, which is a function of third and higher expected moments of pd_t^*.

Note that, by definition, PD_t^* is equal to

$$PD_t^* = M_{t+1}\gamma_{t+1}\left(1 + PD_{t+1}^*\right) \tag{7}$$

Taking the logarithm of this equation yields

$$pd_t^* = m_{t+1} + \Delta d_{t+1} + \ln\left(1 + PD_{t+1}^*\right) \tag{8}$$

where $m_{t+1} = \ln M_{t+1}$, $\Delta d_{t+1} = d_{t+1} - d_t$ and $d_t = \ln D_t$. This expression can be linearized with a first-order Taylor approximation. As shown by Campbell, Lo, and MacKinlay (1997), the last term of equation (8) can be approximated by (cf. Appendix A.2)

$$\ln\left(1 + PD_{t+1}^*\right) \simeq \kappa + \rho pd_{t+1}^* \tag{9}$$

where $\rho = 1/\left(1 + \exp\left(-\overline{pd}_t\right)\right)$ and $\kappa = -\ln\rho - (1-\rho)\ln\left(1/\rho - 1\right)$ are both linearization coefficients and $\overline{pd}_t$ is the average log PD ratio observed until time t.[9] Substituting this approximation in equation (8) gives

$$pd_t^* \simeq \kappa + m_{t+1} + \Delta d_{t+1} + \rho pd_{t+1}^* \tag{10}$$

Finally, by substituting pd_{t+1}^* forward, we get the following linear approximation of pd_t^*

$$pd_t^* \simeq \sum_{i=1}^{\infty} \rho^{i-1} \left(\kappa + m_{t+i} + \Delta d_{t+i}\right) \tag{11}$$

If we now use the approximation (11) in equation (6), we find that

$$pd_t = E_t\left(\sum_{i=1}^{\infty} \rho^{i-1} \left(m_{t+i} + \Delta d_{t+i}\right)\right) + \frac{1}{2} V_t\left(\sum_{i=1}^{\infty} \rho^{i-1} \left(m_{t+i} + \Delta d_{t+i}\right)\right) + c_t' \tag{12}$$

where $c_t' = R_t + \kappa / (1-\rho)$. The next step is to replace the log SDF m_{t+i} by its definition given in equation (3)[10] and to express the fundamental log PD ratio in vector terms to simplify notation. For that, let us define the vector $x_t = \begin{bmatrix} \Delta c_t \ \Delta d_t \end{bmatrix}'$ which collects all the fundamentals . Using this notation, we can rewrite equation (12) as

$$pd_t = \sum_{i=1}^{\infty} \rho^{i-1} g' E_t\left(x_{t+i}\right) + \frac{1}{2} V_t\left(\sum_{i=1}^{\infty} \rho^{i-1} g' x_{t+i}\right) + c_t \tag{13}$$

where $g' = \begin{bmatrix} -\gamma \ 1 \end{bmatrix}$ and $c_t = R_t + (\kappa + \ln \beta) / (1-\rho)$. Developing the conditional variance in the second term of the right-hand side of this equation yields

$$pd_t = \sum_{i=1}^{\infty} \rho^{i-1} g' E_t\left(x_{t+i}\right) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho^{i+j-2} g' E_t\left(\Lambda_{t+i,t+j}\right) g + c_t \tag{14}$$

where $\Lambda_{t+i,t+j} = \varepsilon_{t,t+i} \varepsilon_{t,t+j}'$ and $\varepsilon_{t,t+i} = x_{t+i} - E_t\left(x_{t+i}\right)$ is the error of a forecast made at time t about the fundamental x_{t+i}. The expression $E_t\left(\Lambda_{t+i,t+j}\right)$ is the difference of the expected conditional covariance *at time* t between the $t+i$ and $t+j$ forecast errors.[11] Equation (14) simply states that the fundamental log PD ratio is a linear function of expected fundamentals (dividend growth

[9] Note that the linearization coefficient ρ is time-varying since it changes each time the observed average log PD ratio $\overline{pd}_t$ changes. To keep notation simple, I did not use a subscript for the time with ρ, but the reader should keep in mind that ρ is reestimated at each period to compute the fundamental price. The reestimation of ρ is necessary to arrive at a fundamental price that is computed only on the data observable at time t.

[10] Note that $m_{t+1} = \ln M_{t+1} = \ln \beta - \gamma \Delta c_{t+1}$ where $\Delta c_{t+1} = c_{t+1} - c_t$ with $c_t = \ln C_t$.

[11] In this paper, the term covariance refers both to variances and covariances.

and consumption growth) and of their expected conditional covariances and autocovariances.

2.4 Expectations about future fundamentals

As stated in equation (14), the fundamental log PD ratio is a function of the representative agent's expectations about fundamentals and their conditional covariance. Therefore, to estimate the fundamental price, we have to specify how the representative agent forms her expectation. According to Definition 1, the representative agent is rational, which implies, by definition, that she will use all the relevant information available at time t to make her forecasts. At time t, the relevant information set is constituted of all present and past fundamentals x_t and of all present and past variables, which have some forecasting power for the fundamentals (e.g. inflation). The latter variables are collected in the $(p \times 1)$ vector z_t. Note that z_t can include the observed log PD ratio if it helps to predict future fundamentals. Additionally, we assume the following:

Assumption 2 *The representative agent forms her expectations about future fundamentals in two steps. In the first step, she estimates the dynamic of the variables in her information set with a VAR model, in which the conditional covariance is modeled with a multivariate GARCH. In the second step, she uses the estimated VAR-GARCH to forecast future fundamentals and their conditional covariance.*

The VAR part of the model is used to forecast the first part of the right-hand side of equation (14) (future fundamentals). The GARCH part estimates the dynamic of the conditional covariance, which is then used to forecast the second part of the right-hand side of equation (14) (future conditional covariance). Concretely, assumption 2 implies that the representative agent uses the following model to make her forecasts about future fundamentals:

$$y_t = A_0 + A_1 y_{t-1} + ... + A_j y_{t-j} + \varepsilon_t \tag{15}$$

$$\varepsilon_t \sim N\left(0, H_t\right) \tag{16}$$

where $y_t = \begin{bmatrix} x_t \ z_t \end{bmatrix}'$ is a vector collecting present and past observations (until lag j), A_i are matrices of coefficients[12] estimated at time t and ε_t is an error

[12] The coefficient matrices are time-varying since a rational agent updates her estimation with each new observation. To simplify the notation, I did not use any subscript for t. The reader should, however, bear in mind that a different VAR-GARCH is estimated for each period using only the information available at that time.

term, which is normally distributed with a time-varying covariance matrix H_t. Assumption 2 also specifies that the covariance matrix H_t is modelled as a multivariate GARCH. The multivariate GARCH is a generalization of the univariate GARCH which estimates time-varying covariances in addition to time-varying variances. A general formulation of the multivariate GARCH is the vech model of Bollerslev, Engle, and Wooldridge (1988), which has the following specification

$$h_t = K + Bh_{t-1} + Ce_{t-1} \tag{17}$$

$$e_t = h_t + u_t \tag{18}$$

where h_t =vech$\{H_t\}$, e_t =vech$\{\varepsilon_t \varepsilon_t'\}$ where the vech operator converts the lower triangle of a symmetric matrix into a vector. The matrices K, B and C are matrices of coefficients[13] and u_t is an error term which is normally distributed with a constant covariance matrix Ω. In this model, the covariance matrix is a linear function of its last past values and of last past residuals.[14] For greater clarity, it is useful to express this VAR-GARCH model as its companion form

$$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{e}_t \tag{19}$$

$$\mathbf{h}_t = \mathbf{B}\mathbf{h}_{t-1} + \mathbf{u}_t \tag{20}$$

where

$$\mathbf{y}_t = \begin{bmatrix} y_t \; y_{t-1} \; ... \; y_{t-j+1} \; 1 \end{bmatrix}' \quad \mathbf{h}_t = \begin{bmatrix} h_t \; e_t \; 1 \end{bmatrix}'$$

$$\mathbf{e}_t = \begin{bmatrix} \varepsilon_t \; 0 \; ... \; 0 \end{bmatrix}' \quad \mathbf{u}_t = \begin{bmatrix} 0 \; u_t \; 0 \end{bmatrix}'$$

$$\mathbf{A} = \begin{bmatrix} A_1 & \cdots & A_{j-1} & A_j & A_0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} B & C & K \\ B & C & K \\ 0 & 0 & 1 \end{bmatrix}$$

If the representative agent uses this VAR-GARCH model to make her forecasts, her expectation about $\mathbf{y}_{t+i}$ and H_{t+i} will be

$$E_t\left(\mathbf{y}_{t+i}\right) = \mathbf{A}^i \mathbf{y}_t \tag{21}$$

[13] The coefficient matrices are time-varying (cf. footnote 12).

[14] In the original vech model, the correlation matrix can be a function of more than one lag (and of other exogeneous variables). For simplicity, however, and also because it will be formally expressed in assumption 5, I restrict the model to one lag (with no exogeneous variables).

$$E_t\left(H_{t+i}\right) = \text{vech}^{-1}\left\{\mathbf{B}^i\mathbf{h}_t\right\} \tag{22}$$

where the operator vech^{-1} converts a vector into the lower triangle of a symmetric matrix.

Finally, before using these expectations in the fundamental equation (14), it is useful to define $\mathbf{H}_{t+i} = \mathbf{e}_{t+i}\mathbf{e}'_{t+i}$. We find that

$$E_t\left(\mathbf{H}_{t+i}\right) = E_t\left(\mathbf{e}_{t+i}\mathbf{e}'_{t+i}\right) = \begin{bmatrix} E_t\left(H_{t+i}\right) & 0 \ldots 0 \\ 0 & 0 \ldots 0 \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \ldots 0 \end{bmatrix} =$$

$$= \mathbf{q}\text{vech}^{-1}\left\{\mathbf{B}^i\mathbf{h}_t\right\}\mathbf{q}' \tag{23}$$

where $\mathbf{q}' = \left[\mathbf{I}_{2+p}\ \mathbf{0}_{(2+p)\times(2+p)(j-1)}\right]$.

2.5 Fundamental PD ratio

Once the expectations have been defined, it is possible to derive the fundamental log PD ratio as a function of the estimated VAR-GARCH and of the observable variables. For that, let us rewrite equation (14) with the new notation:

$$pd_t = \sum_{i=1}^{\infty}\rho^{i-1}\mathbf{g}'E_t\left(\mathbf{y}_{t+i}\right) + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho^{i+j-2}\mathbf{g}'E_t\left(\mathbf{\Lambda}_{t+i,t+j}\right)\mathbf{g} + c \tag{24}$$

where $\mathbf{g}' = \left[g\ \mathbf{0}_{(2+p)(j-1)+p}\right]$ and $\mathbf{\Lambda}_{t+i,t+j} = \mathbf{e}_{t,t+i}\mathbf{e}'_{t,t+j}$ with $\mathbf{e}_{t,t+i} = \left[\varepsilon_{t,t+i}\ 0\ \ldots\ 0\right]'$. The vector $\mathbf{g}'$ is a row vector which selects x_t in $\mathbf{y}_t$ and multiplies it by the vector g. [15]

Let us first consider the second part of the right-hand side of equation (24). Using the fact that $\mathbf{y}_{t+1} = \mathbf{A}_t\mathbf{y}_t + \mathbf{e}_{t+1}$, we can express the error made at time t about the vector $\mathbf{y}_{t+i}$ as a function of one-period shocks

$$\mathbf{e}_{t,t+i} = \sum_{k=1}^{i}\mathbf{A}^{i-k}\mathbf{e}_{t+k} \tag{25}$$

[15] Note that, given assumption 2, expected third and higher moments are constant. Therefore, the remainder R_t of the Taylor expansion, and thus c_t, are also constant.

Plugging that into $\mathbf{\Lambda}_{t+i,t+j}$ yields

$$\mathbf{\Lambda}_{t+i,t+j} = \sum_{k=1}^{i} \sum_{l=1}^{j} \mathbf{A}_t^{i-k} E_t \left(\mathbf{e}_{t+k} \mathbf{e}'_{t+l} \right) \left(\mathbf{A}'_t \right)^{j-l} \tag{26}$$

Using this expression and the fact that the error terms are not autocorrelated,[16] we can rewrite equation (24) as

$$\begin{aligned} pd_t = & \sum_{i=1}^{\infty} \rho^{i-1} \mathbf{g}' E_t \left(\mathbf{y}_{t+i} \right) + \\ & + \frac{1}{2} \sum_{i=1}^{\infty} \rho^{i-1} \mathbf{g}' \left(\mathbf{I} - \rho \mathbf{A} \right)^{-1} E_t \left(\mathbf{H}_{t+i} \right) \left(\mathbf{I} - \rho \mathbf{A}' \right)^{-1} \mathbf{g} + c \end{aligned} \tag{27}$$

We can integrate the expectation derived in equations (21) and (23) into our fundamental equation to arrive at

$$\begin{aligned} pd_t = & \sum_{i=1}^{\infty} \rho^{i-1} \mathbf{g}' \mathbf{A}^i \mathbf{y}_t + \\ & + \frac{1}{2} \sum_{i=1}^{\infty} \rho^{i-1} \mathbf{g}' \left(\mathbf{I} - \rho \mathbf{A} \right)^{-1} \mathbf{q} \text{vech}^{-1} \left\{ \mathbf{B}^i \mathbf{h}_t \right\} \mathbf{q}' \left(\mathbf{I} - \rho \mathbf{A}' \right)^{-1} \mathbf{g} + c \end{aligned} \tag{28}$$

Finally, using the fact that the vech^{-1} operator has the following properties

$$a \text{vech}^{-1} (x) = \text{vech}^{-1} (ax) \tag{29}$$

$$\text{vech}^{-1} (x) + \text{vech}^{-1} (y) = \text{vech}^{-1} (x + y) \tag{30}$$

we obtain the last fundamental equation

$$\begin{aligned} pd_t = & \mathbf{g}' \left(\mathbf{I} - \rho \mathbf{A} \right)^{-1} \mathbf{A} \mathbf{y}_t + \\ & + \frac{1}{2} \mathbf{g}' \left(\mathbf{I} - \rho \mathbf{A} \right)^{-1} \mathbf{q} \text{vech}^{-1} \left\{ \left(\mathbf{I} - \rho \mathbf{B} \right)^{-1} \mathbf{B} \mathbf{h}_t \right\} \mathbf{q}' \left(\mathbf{I} - \rho \mathbf{A}' \right)^{-1} \mathbf{g} + \\ & + c \end{aligned} \tag{31}$$

Equation (31) expresses the fundamental log PD ratio as a function of 1) the observable variables in $\mathbf{y}_t$, 2) the estimated covariance matrix in $\mathbf{h}_t$, 3) the estimated VAR-GARCH dynamic in $\mathbf{A}$ and $\mathbf{B}$, 4) the coefficient of relative risk aversion γ in $\mathbf{g}$ and 5) the linearization parameter ρ. The next sections present the empirical estimation of this fundamental equation.

[16] Cf. equation (16).

3 Data

The next assumption concerns the set of variables y_t that is used to forecast future fundamentals (i.e. future consumption and dividends).

Assumption 3 *The representative agent uses past and present fundamentals (i.e. consumption and dividends) and inflation rates to forecast future fundamentals.*

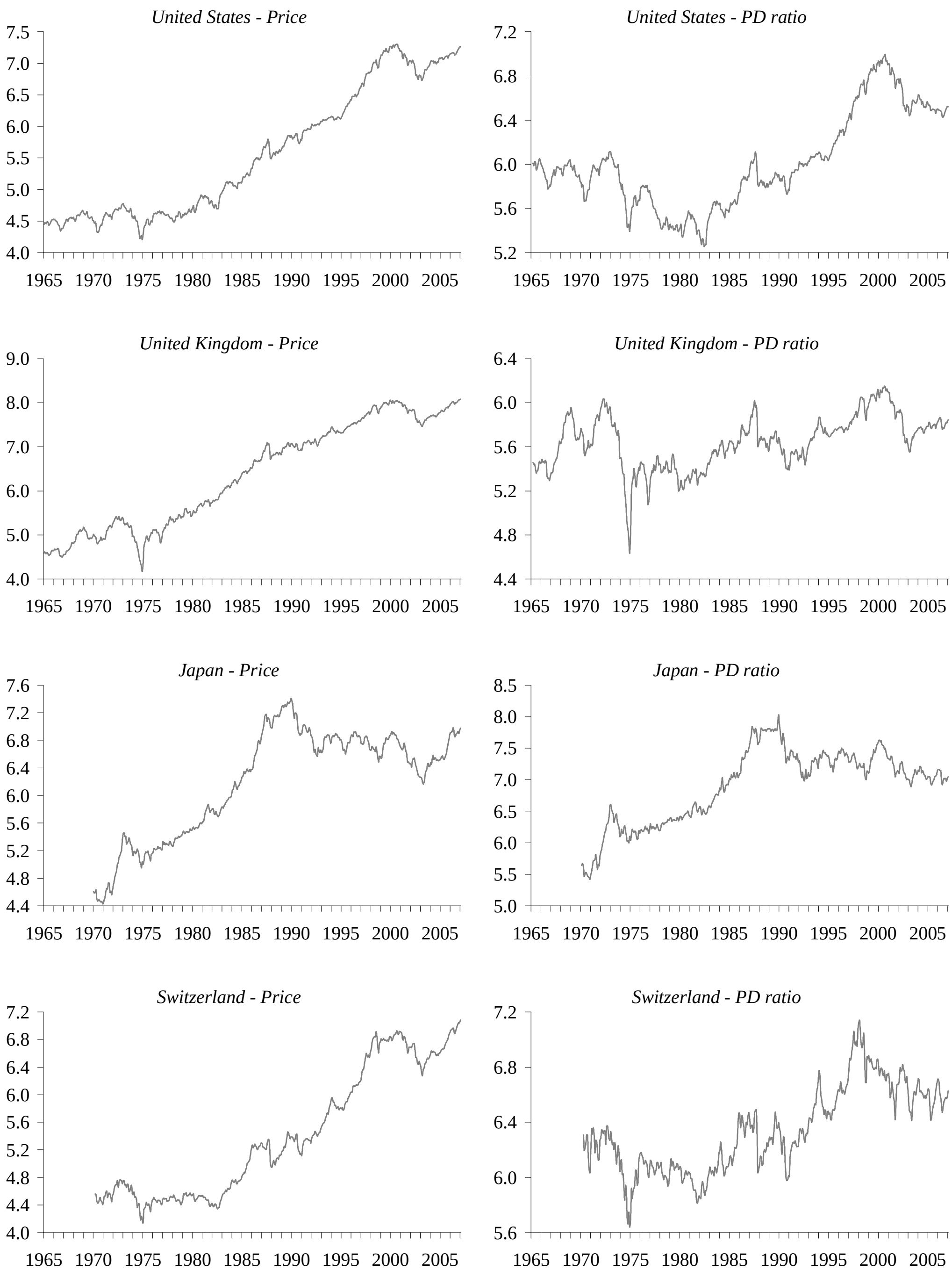
Following assumption 3, the data set contains stock prices, dividends, consumption and consumer price data for four countries: United States, United Kingdom, Japan and Switzerland. All series are monthly. Stock prices are measured by the S&P 500 index for United States, the FTSE All shares index for Untied Kingdom and the MSCI indexes for Japan and Switzerland. Monthly data are obtained by taking the monthly average of daily prices. Dividends are computed with the dividend yield data supplied for each stock price index. Consumption is measured by personal consumption expenditures in the United States and by the households' consumption expenditure for the other countries. Consumption data for the United Kingdom, Japan and Switzerland are quarterly. They have been converted to monthly data by using the Eviews cubic spline conversion method.[17] Whenever necessary, real values are obtained by dividing nominal data by the Consumer Price Index of each country. The samples extend from January 1965 through to January 2007 for the United States and the United Kingdom, from January 1970 through to January 2007 for Japan and from March 1970 through to January 2007 for Switzerland. All data stem from Datastream except consumption data for Japan and Switzerland, which stem from the IMF database and the Swiss National Bank database respectively. Figure 1 displays the log nominal stock price indexes and the log PD ratios. Figure 2 shows the log nominal consumption and the log nominal dividends. Finally, Figure 3 gives the log consumer price levels.

4 Long-term evolution of PD ratios and of inflation

PD ratios in the United States, the United Kingdom, Japan and Switzerland share a striking feature: they all increased significantly during the observation period (cf. Figure 4). The visual impression given by Figure 4 is formally confirmed by structural break tests. Table 1 gives the results of the sup$F$

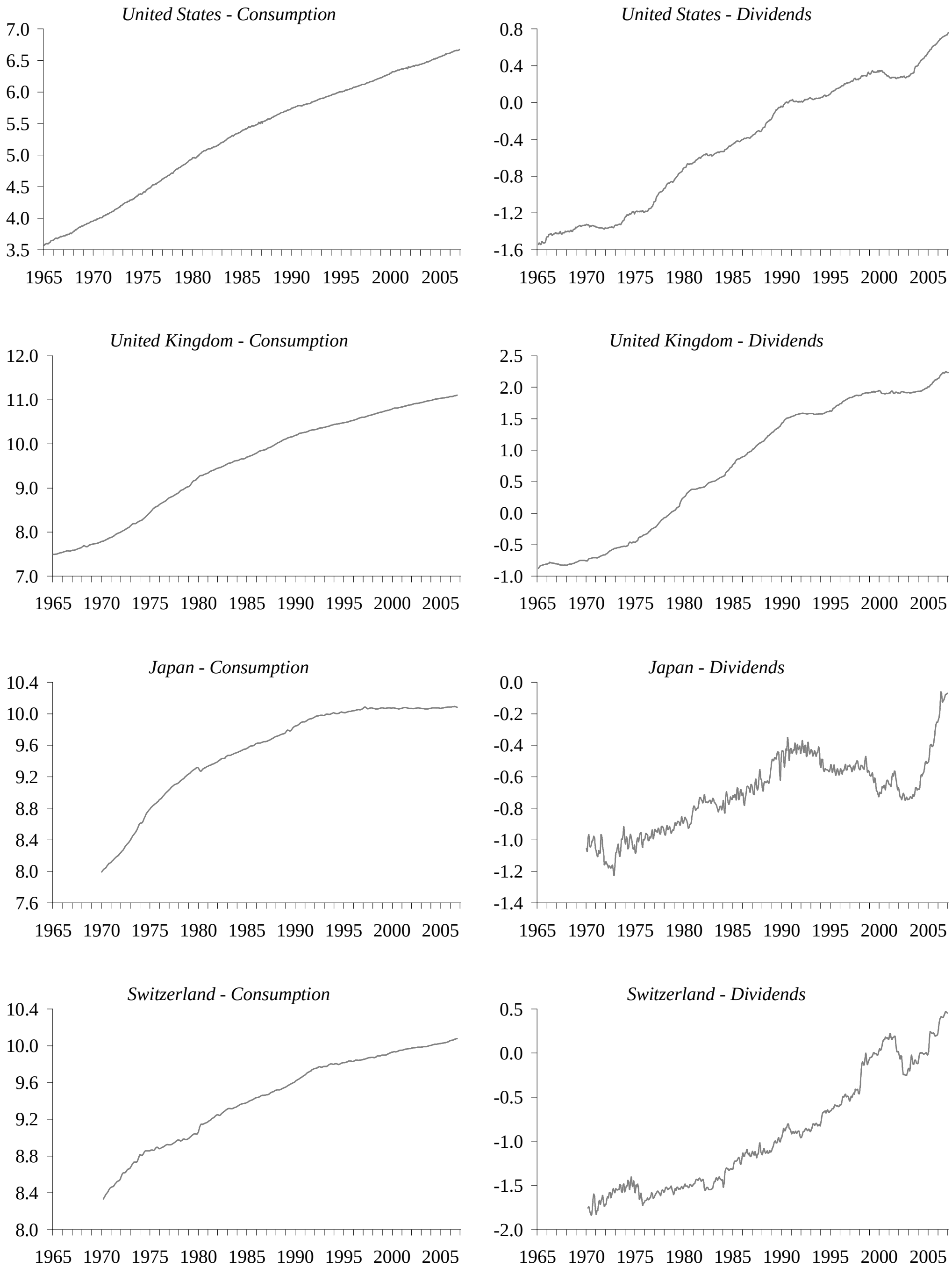
[17] This method assigns each value in the low frequency series to the last high frequency observation associated with the low frequency period, then places all intermediate points on a natural cubic spline connecting all the points.

Fig. 1. Stock prices and PD ratios



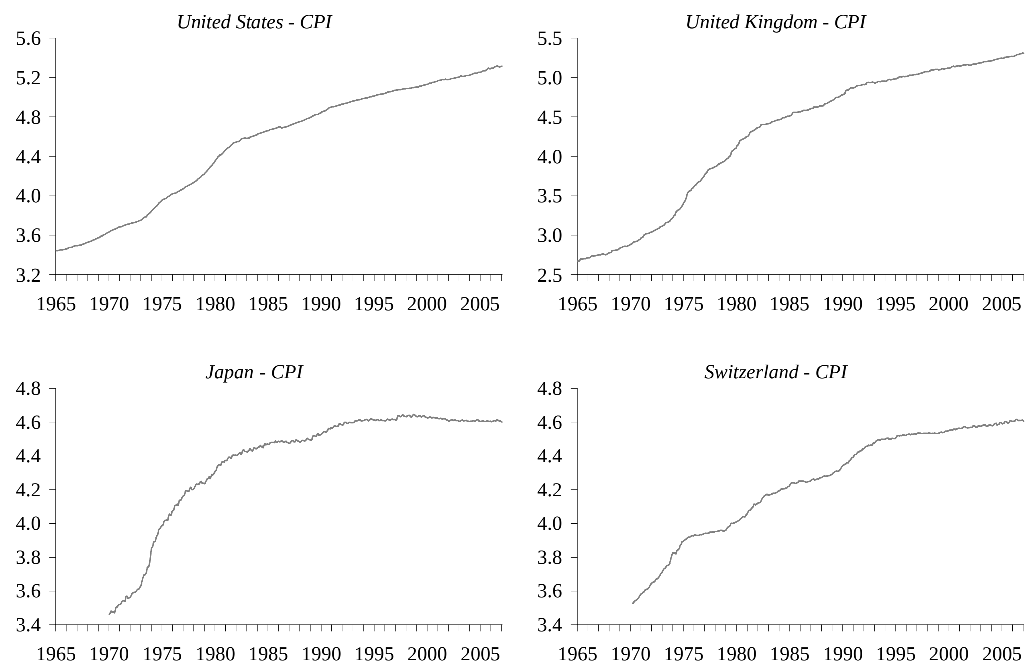
Price indexes: S&P 500 for the Untied States, FTSE All shares for the United Kingdom, MSCI for Japan and Switzerland. Dividends are computed with the dividend yield associated with each of these indexes. Prices are nominal. Both variables are expressed in logarithm. Source: Datastream.

Fig. 2. Consumption and dividends



Consumption: personal consumption expenditures for the United States and households' final consumption for the United Kingdom, Japan and Switzerland. Dividends are computed with the dividend yield associated with each stock price index. Both variables are nominal and expressed in logarithm. Source: Datastream, IMF and Swiss national bank.

Fig. 3. Consumer price indexes



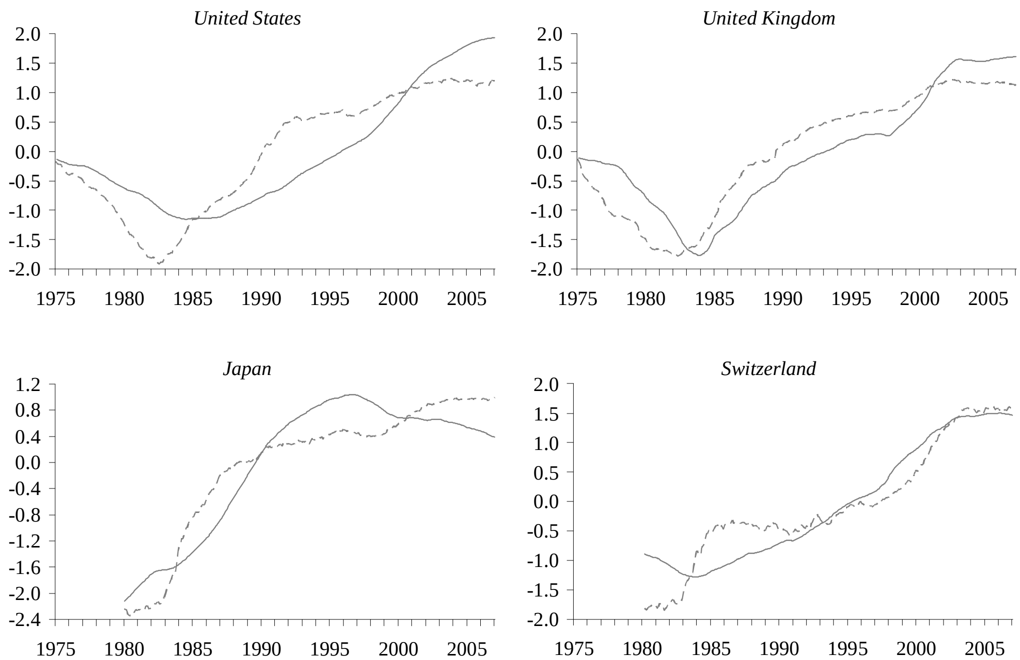
Consumer price indexes are expressed in logarithm. Source: Datastream.

structural break test developed by Bai and Perron (1998, 2003a) for the log PD ratio in the different countries. This test assesses the probability of a structural break in the average PD ratio and estimates its most probable date. The Bai and Perron's sup$F$ statistics clearly detect a structural break in PD ratios for each country. These breaks occured in the first half of the 1990s in the United States, the United Kingdom and Switzerland and around 1983 in Japan. In all countries, the average PD ratio increases significantly after the break.

The four countries studied in this paper share another characteristic: they all experienced a significant decline in inflation in the 1980s. This decline could be an explanation for the higher PD ratio. Several studies show that the yield on stocks is linked to the yield on nominal Treasury bonds. This link is the center of the so-called "Fed model", which is quite successful in empirically describing stock prices. Historically, the main influence on the nominal bond yield is inflation. Thus, inflation should have an indirect impact on stock prices through its influence on nominal bond yields. Indeed, in our sample, the long-term evolution of the PD ratio is strikingly similar to that of inflation rates (cf. Figure 4).

The literature usually distinguishes three channels through which inflation can influence PD ratios. First, inflation and the response of monetary policy

Fig. 4. Long-term evolution of PD ratios and of inflation



Each panel gives the 10 – year moving average of the PD ratio and the inflation (solid and dashed line, respectively). Both variables are standardized.

to it can damage the real economy. This can impair the profitability of firms and thus reduce dividends. In that case, a fall in inflation is beneficial to the firms, which can then distribute more dividends, driving up PD ratios. Second, higher inflation can induce more uncertainty in the economy, leading to higher risk premiums. In this second case, the lower inflation observed since the eighties is associated with lower risk premiums and thus higher PD ratios. Finally, a third, and more radical, possibility could be that investors fail to understand the effect of inflation on nominal variables. This phenomenon is called money illusion (or inflation illusion). Fisher (1928) gives the following definition for money illusion: [18]

> "[Money illusion] is the failure to perceive that the dollar, or any other unit of money, expands or shrinks in value"

If an investor suffers from money illusion, then she does not take inflation into account (or does so falsely). One consequence of this is that she might value an equity by using nominal dividends instead of real dividends (Modigliani and Cohn, 1979). Another consequence is that she might measure her consumption in nominal rather than in real terms. This would bias her discount factor, since she will use a nominal discount rate instead of a real one.

[18] Citation taken from Brunnermeier and Julliard (2008).

Table 1
Structural breaks

		1 break vs. 0 break	Date	Direction of change
Unites Stated	PD ratio	370.9814**	1995.10	+
	Inflation	109.7876**	1982.08	-
United Kingdom	PD ratio	170.6247**	1993.09	+
	Inflation	82.5391**	1982.06	-
Japan	PD ratio	339.9369**	1983.08	+
	Inflation	79.5824**	1977.05	-
Switzerland	PD ratio	310.4308**	1993.01	+
	Inflation	92.6971**	1975.11	-

* (**) denotes the rejection of the null hypothesis of 0 breaks for the alternative hypothesis of 1 break at a 5% (1%) confidence level. The critical values for the test are given by Bai and Perron (2003b) for a trimming parameter of 0.15. The estimated break date is given in the last column.

The first two channels (i.e. impact on dividend growth rates and on risk premiums) are compatible with a rational investor framework. The last one (i.e. money illusion) assumes some level of irrationality from the investors in the sense that they fail to correctly interpret the impact of inflation on the economy. In this paper, I study these two situations separately and compare them with a situation in which inflation has no structural break. Thus, in the rest of this paper, I compare the three following models:

Model 1 (No break) *Inflation has an influence on dividend growth rate and risk premiums but does* not *display structural breaks*

Model 2 (With breaks) *Inflation has an influence on dividend growth rate and risk premiums and does display structural breaks*

Model 3 (With money illusion) *Agents are subject to money illusion and use nominal instead of real variables*

The last two models imply that agents have to react to structural breaks (in inflation with model 2 and in nominal variables with model 3). To estimate how investors integrate structural breaks in their forecasts, I make the following assumption:

Assumption 4 *The representative agent bases her forecasts about future growth rates of consumption, dividends or consumer prices on the average growth rate observed in the last five*[19] *years.*

By taking the average growth rate over the last years instead of the average over the whole period, the agent obtains an estimate of the long-term growth rate which is *currently* driving the variables. Thus she is able to adapt her expectation to a structural change in any of the variables. Assumption 4 does not mean that the agent abandons the use the VAR-GARCH model, but rather that she uses it to forecast deviations from the long-term average. Note that I do not use Assumption 4 to compute the fundamental price with model 1 since variables are assumed to be stable in that case.

To take into account the impact of a structural decrease in inflation on dividends (on the risk premium), I regress inflation on the dividend growth rate (on the consumption growth rate) at each period and then use the coefficients obtained to estimate the impact of a change in average inflation on the average dividend growth rate (consumption growth rate). Finally, to estimate the effect of money illusion, I use nominal dividend and consumption growth rates instead of real growth rates (as an agent subject to money illusion is assumed to do). The difference between the real (with break in inflation) and the nominal model (with break in nominal variables) reflects the impact of money illusion.

5 Econometric methodology

To be able to compute the fundamental price given by equation (31), we must know $\mathbf{A}$, $\mathbf{B}$, ρ and c. The next two sections explain how to estimate these coefficients.

5.1 Estimation of the VAR-GARCH model

The first step of the computation of the fundamental stock price is to estimate the matrices $\mathbf{A}$ and $\mathbf{B}$ of the VAR-GARCH model. This requires the simultaneous estimation of equations (15) and (17). For the estimation of the GARCH dynamic to remain computationally feasible, it is necessary to restrict the number of coefficients in $\mathbf{A}$ and $\mathbf{B}$. For that, I make the following assumption:

[19] I also estimated the fundamental prices with an average over the last 7.5 and 10 years. In the text, I present only the alternative which gives the best results in terms of forecasts.

Assumption 5 *Each element of the conditional covariance matrix depends only on its own last value and last residuals.*

This assumption means that the GARCH dynamic is a BEKK model (Engle and Kroner, 1995) with one lag and no exogenous variable. The BEKK model can be estimated by maximization of the log likelihood function as explained by Hamilton (1994, p. 670).

I estimated a VAR-GARCH model for each period using only the data available at that time. With this procedure, I placed myself exactly in the same situation as an investor living at time t. This yields a truly "out-of-sample" estimation of the fundamental price, i.e. based only on *ex-ante* data. In each period I used 6 lags in the VAR part of the model. In addition to the matrices $\mathbf{A}$ and $\mathbf{B}$, the estimation of the VAR-GARCH also gives an estimation of the conditional variance-covariance vector $\mathbf{h}_t$.

5.2 Estimation of the relative risk aversion coefficient

After the estimation of the matrices $\mathbf{A}$ and $\mathbf{B}$, the only unknown remaining in the fundamental equation (31) is the relative risk aversion coefficient γ appearing in $\mathbf{g}$. Note that, without using the matrix notation, equation (31) is equivalent to

$$pd_t = -\gamma q_{1,t} + q_{2,t} + \frac{1}{2}\gamma^2 q_{3,t} - \gamma q_{4,t} + \frac{1}{2} q_{5,t} + c \tag{32}$$

where $q_{1,t}$ and $q_{2,t}$ are the first and second elements of $(\mathbf{I} - \rho\mathbf{A})^{-1}\,\mathbf{A}\mathbf{y}_t$ for $\mathbf{A}$ and ρ estimated with the sample available at time t, and $q_{3,t}$, $q_{4,t}$ and $q_{5,t}$ are the (1,1)-th, (1,2)-th and (2,2)-th element of

$$(\mathbf{I} - \rho\mathbf{A})^{-1}\,\mathbf{q}vech^{-1}\left\{(\mathbf{I} - \rho\mathbf{B})^{-1}\,\mathbf{B}\mathbf{h}_t\right\}\mathbf{q}'\,(\mathbf{I} - \rho\mathbf{A}')^{-1} \tag{33}$$

for $\mathbf{A}$, $\mathbf{B}$, ρ and $\mathbf{h}_t$ estimated with the sample available at time t. The parameters γ and c can then be estimated, for each period, by estimating equation (32) with OLS and a sample containing only the variables $q_{\cdot,t-k}$ (for all k between 0 and $t-1$) available at this time.

6 Estimated fundamental stock prices

The fundamental log PD ratio can be computed with equation (31) for parameters estimated as explained in section 5. As mentioned in section 4, I use real data for the models without money illusion and nominal data for the models with money illusion. The fundamental price can then be recovered from the fundamental log PD ratio by adding d_t (in real terms for the models without money illusion and in nominal terms for the model with money illusion). Figure 5 shows the estimated fundamental stock prices for the different models described in section 4 and the observed prices (left panels) and the estimated fundamental and observed PD ratios (right panels). The gap between the observed and the fundamental prices is presented in Figure 6.

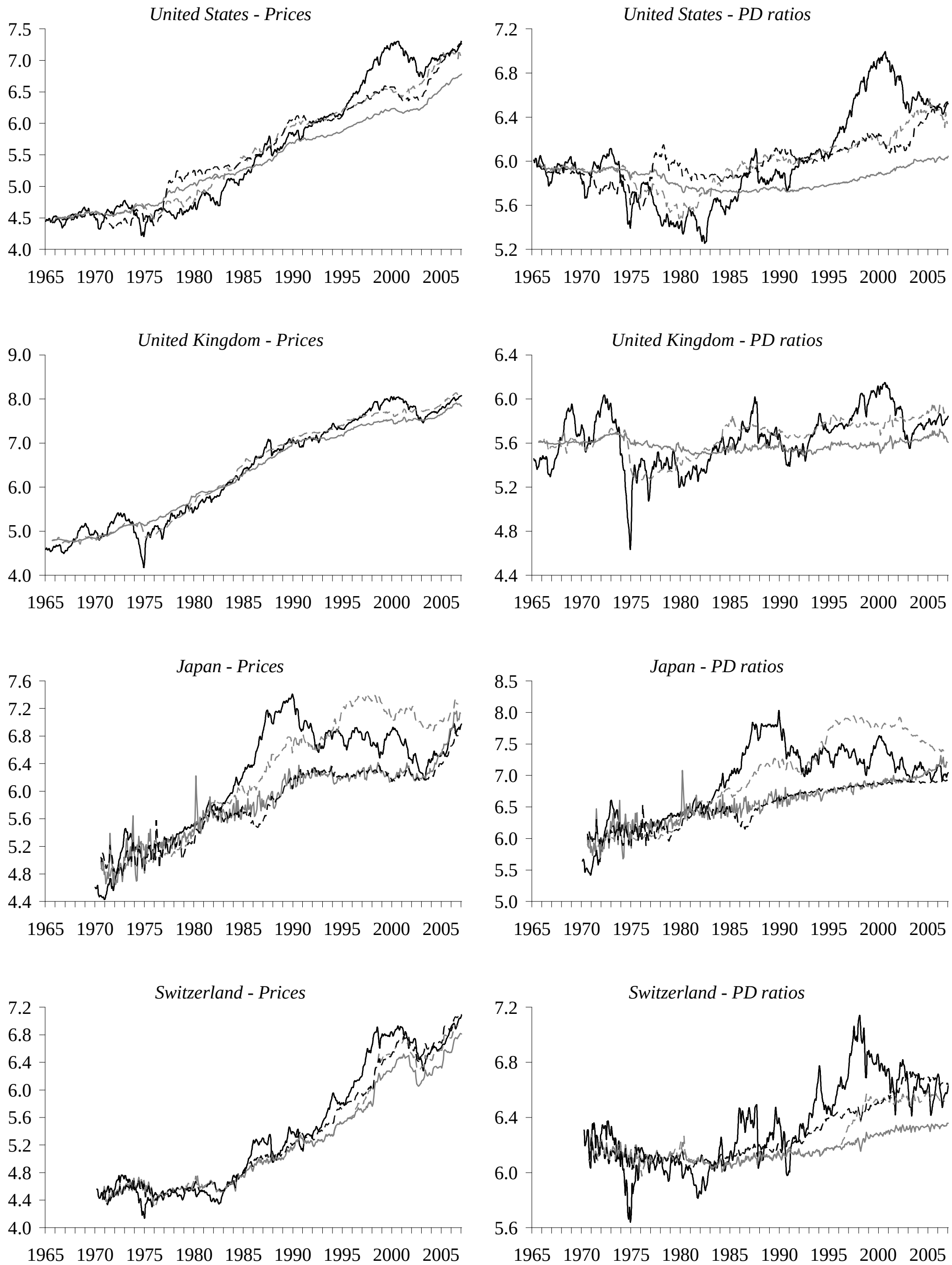
In each country, we observe that the price can diverge significantly and for long periods from its fundamental value for all three models. The American stock market is characterized by a huge gap at the end of the 1990s, which is associated with the Internet bubble. According to the model with breaks in inflation and to the model with money illusion, this bubble has disappeared in recent years. Before the bubble, and according to all models, the stock prices were undervalued most of the time, with some short episodes of overvaluation (e.g. in 1974 or in 1987).[20] The stock market in the United Kingdom follows a similar pattern, with a smaller overvaluation at the end of the 1990s (and a bigger undervaluation after the crash in 1974). Note that for this country I do not observe a significant difference between the models with and without breaks in inflation, which suggests that no strong empirical link between inflation and real variables has been observed during this period. The Japanese market saw a long period of relatively substantial overvaluation at the end of the 1980s and at the very beginning of the 1990s. After that, the model with money illusion differs significantly from the others. The model with money illusion indicates that stocks were undervalued whereas the other models indicate that stocks were overvalued until 2004. Finally, the Swiss market displays a pattern similar to the United Kingdom with the exceptions that the main overvaluation period begins earlier in the 1990s and decreases most notably in 1998. Note that the Swiss market suffered from the 1998 crisis (Russian/LTCM crisis) more than other western countries. July 1998 remained the historical peak until the beginning of 2006 in Switzerland, whereas stock prices recovered much more rapidly in other western countries.[21]

The volatility of the observed price is significantly greater than the volatility of

[20] Zhong, Darrat, and Anderson (2003) and Black, Fraser, and Groenewold (2003b) find a similar period of undervaluation between the second half of the 1970s and the first half of the 1990s. The former use *ex-post* data for a sample ranging from 1871 through to 1997 (as in Shiller, 1981); the latter use a general equilibrium framework with samples ranging from 1947 through to 2002 for quarterly data and from 1929 through to 2001 for annual data.

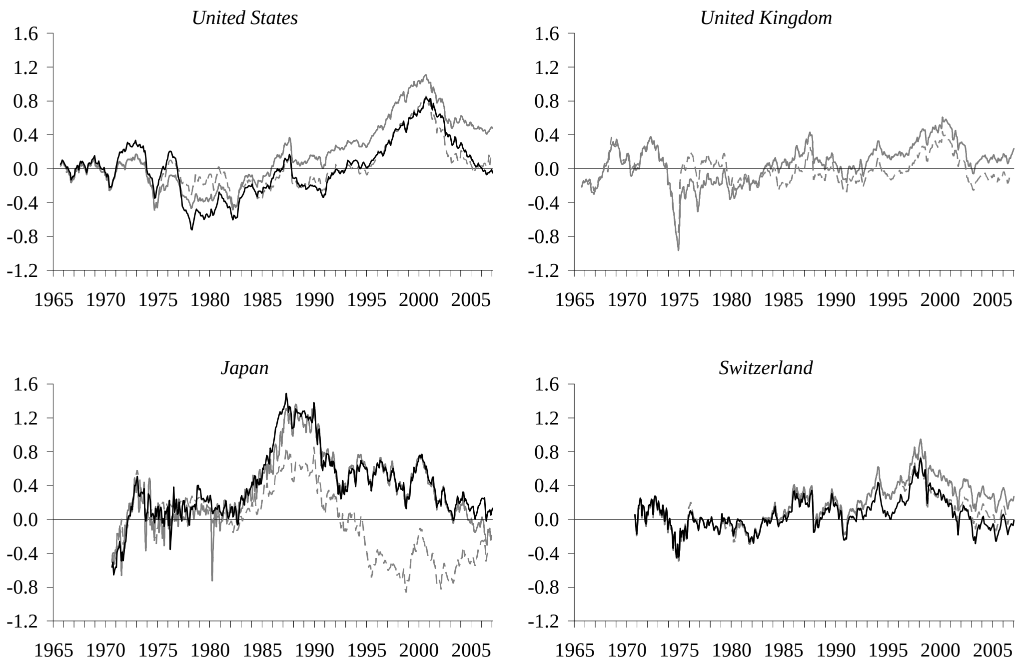
[21] More precisely, for the Swiss market, the record high of 21 July 1998 (1931.49) was beaten for one day (23 August 2000) before plunging until the beginning of 2003. On the other hand, the stock price index returned to its level of July 1998 only five months later in the United States and ten months later in the United Kingdom.

Fig. 5. Fundamental stock prices and fundamental PD ratios



The black line is the observed stock price. The grey line is the estimated fundamental stock price without break. The black dashed line is the estimated fundamental stock price with breaks in inflation. The grey dashed line is the estimated fundamental stock price with money illusion. All variables are expressed in logarithm.

Fig. 6. Gap between the observed and the fundamental prices



Each panel presents the difference between the (log) observed nominal stock price and the estimated (log) fundamental stock price without break (grey line), with break in inflation (black line) and with money illusion (dashed line). For the United Kingdom, the gap for the model with break in inflation is not represented because it is almost similar to the model without break.

the fundamental price without structural breaks in all countries. This is in-line with the stock price volatility puzzle (Shiller, 1981, LeRoy and Porter, 1981). The volatility of the observed price is also significantly greater than the volatility of the fundamental price with breaks in inflation in all countries. Finally, the volatility of the stock price is significantly greater than the fundamental stock price with money illusion in the United States and in Switzerland. In the United Kingdom, this volatility is also greater but the difference is not significant. In Japan, the volatility of the stock price is smaller than that of the fundamental price with money illusion.

7 Do stock prices go back to their fundamental value?

As shown in the previous section, prices can diverge significantly from their fundamental value for long periods. This conclusion naturally raises the question of the existence (or absence) of a link between observed and fundamental prices. I study this question from two points of view. Firstly, I examine the dynamic of the gaps and determine whether they tend to disappear after a certain time. Secondly, I seek to establish whether the fundamental price can help in forecasting future prices out-of-sample. If the fundamental price yields good out-of-sample forecasts, then it is a sign that there is a link between them. The first approach is in-sample and the second is out-of-sample.

7.1 Gap dynamic analysis

Firstly, I test for the presence of a unit root in the gap between the observed and the fundamental prices. If the unit root is rejected, then the gap is mean reverting, which means that it tends to disappear after some time and that the observed price eventually goes back to its fundamental value.[22]

Table 2 presents the results of the Augmented Dickey-Fuller test and the Phillips-Perron test for unit root. For the United States, a unit root cannot be rejected at a 5% confidence level for the models with and without breaks in inflation. The results are less clear for the model with money illusion. In the United Kingdom, the unit root is rejected for all models. In Japan, the unit root cannot be rejected for the model with breaks in inflation and for the model with money illusion. Results are less clear for the model without break. Finally, in Switzerland, the unit root is strongly rejected for the models with breaks and with money illusion. These results suggest that, firstly, there is no clear link between the observed price and the model with or without breaks in the United States, nor between the model with breaks or with money illusion in Japan. Secondly, the gap for all models in the UK and for the models with breaks or money illusion model in Switzerland are mean reverting, indicating a link between these fundamental prices and the observed prices.

In complement to the unit root tests, I estimate the impulse-response function of the different gaps. The results are presented in Figure 7. Each impulse-response function indicates how the gap evolves after a shock. All functions show that the gap generated by a random shock tends to disappear after some time but at very different speeds. In the United States and the United Kingdom, the gap vanishes more quickly for the model with money illusion. In Japan, the gap disappears the fastest in the case of the model without break. In Switzerland, this holds for the model with breaks in inflation. The half-life[23] of the fastest-vanishing gap is one year and two months in the United Kingdom and one year in Switzerland against about 2.5 years in Japan and more than 4 years in the United States for the model with money illusion. This is a sign that there is a link between the observed and the fundamental

[22] This approach is equivalent to testing for cointegration between the fundamental and the observed prices. In our case, the cointegrating vector is known *a priori* and is equal to $\begin{bmatrix}1 & -1\end{bmatrix}$.

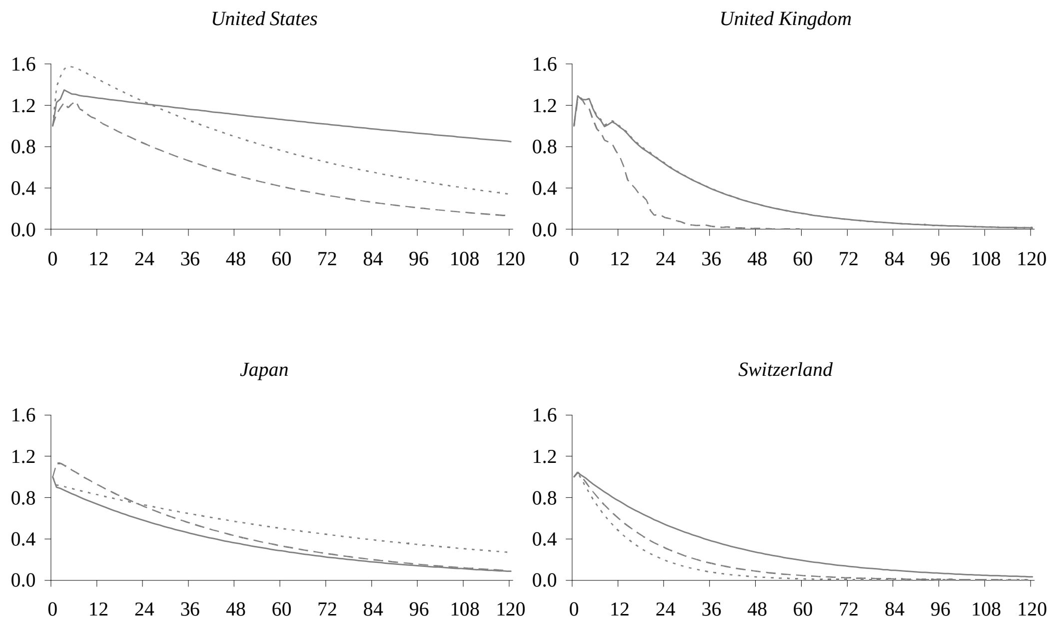
[23] The half-life corresponds to the number of months needed for the gap to wipe out half of the initial shock.

Table 2. Unit root tests for the gaps

	Without break		With breaks		With money illusion	
	ADF test	PP test	ADF test	PP test	ADF test	PP test
United States	-0.8149	-0.8814	-1.7528	-1.8288	-1.9209	-2.0302*
United Kingdom	-2.9596**	-2.2296*	-2.9263**	-2.5002*	-4.2921**	-3.7850**
Japan	-1.3898	-1.9448*	-1.7038	-1.6570	-1.7706	-1.8606
Switzerland	-1.9100	-2.0673*	-3.2116*	-3.5495**	-2.5886**	-2.8800**

* (**) denotes the rejection of the null hypothesis of unit root at a 5% (1%) confidence level. The test is performed with an equation without trend or constant. ADF stands for Augmented Dickey-Fuller and PP for Philipps-Perron.

Fig. 7. Impulse-response function for the gaps



Impulse-Response functions for the gaps between the observed and the fundamental prices (expressed in months, horizontal axis). The solid line corresponds to the fundamental model without break, the doted line to the fundamental model with break in inflation and the dashed line to the fundamental model with money illusion.

prices, but that this link is weak and that it takes time for the gap to die out after a shock. This suggests that the C-CAPM fundamental price does play a role in the *long-term* evolution of the stock price, but that it performs rather poorly in explaining the short-term dynamic of the stock price.

7.2 Forecast accuracy analysis

This section analyses the ability of the information contained in the fundamental price to forecast future stock prices. Apart from its obvious practical applications, the forecast ability of the fundamental price is another way to study the link between the price and its fundamental value. If the fundamental price is able to give good out-of-sample forecasts, then it is a sign that there is a link between them. I compare two forecasting models:

(1) **Random walk with drift (benchmark model)**: for this model, I first estimate the growth rate of the observed price on the observable sample at time t. Then, I use the estimated growth rate to forecast the future price for different horizon starting at the price observed at time t.
(2) **Fundamental random walk with drift**: similarly to the benchmark model, I first estimate the growth rate of the *fundamental* price on the

observable sample at time t. Then, I use the estimated growth rate to forecast the future fundamental price for different horizon starting at the fundamental price at time t. I set the forecast for the future price equal to the forecast for the fundamental price. [24]

The last model has been tested with the fundamental price given by the three models described in section 4. Note that, in addition to these models, I have also estimated models in which the observed price or the fundamental price is a random walk or an AR(1) process. However, their forecasts are constantly outperformed by those of the random walk with drift models. [25] Their results are therefore not presented here. Finally, remember that all forecasts are made out-of-sample. The forecast accuracy is measured by the mean absolute error. [26]

Figure 8 compares the forecast accuracy of the different fundamental models with the benchmark for different forecast horizons (horizontal axis). A value below one indicates that the model is more accurate than the benchmark.

In the United States, the fundamental price *with money illusion* significantly improves forecast accuracy for forecast horizons longer than 2 years. For a horizon of 6 years, the improvement is of 40%. The model with break in inflation also improves forecasts but only for horizons longer that 3.5 years. It is, however, always outperformed by the model with money illusion for a horizon shorter than 5 years and has about the same accuracy thereafter. For the United Kingdom, all models improve forecast accuracy. Models *without money illusion* [27] give the better forecasts for horizons longer than 3 years with an improvement of about 40% in comparison with the benchmark. The model with money illusion gives better forecasts for horizons shorter than 3 years. For Japan, the model *with breaks* also improves forecasts for horizons longer than 5 years. Its performance is, however, less impressive than for the

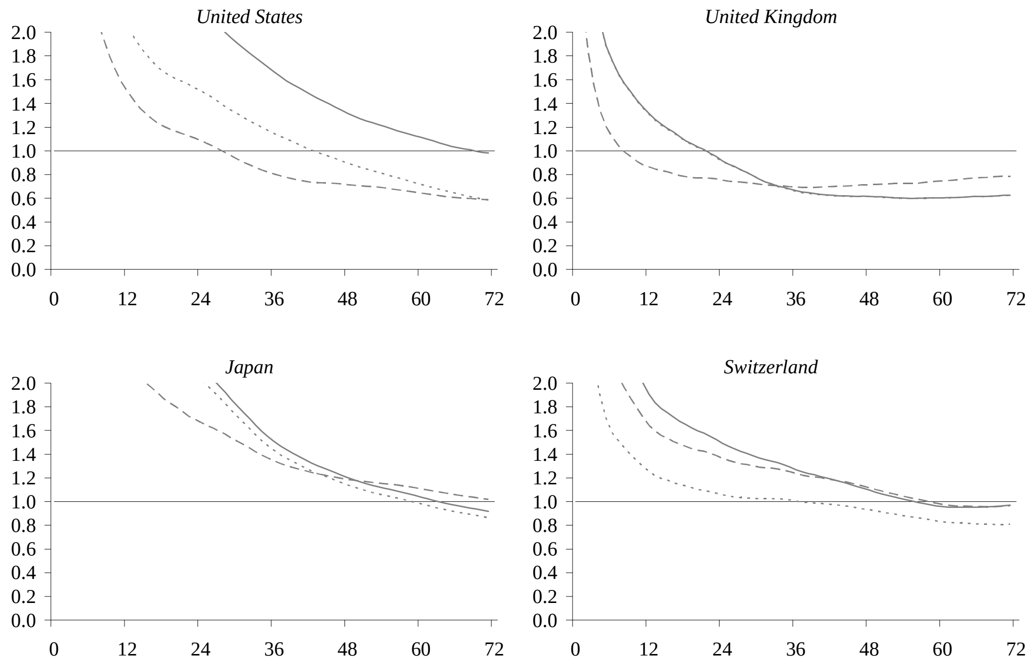
[24] I also used two additional forecasting techniques: one in which I estimate the gap dynamic and combine it with the fundamental random walk model, and one in which I try to predict the error made by the fundamental model with the actual value of the gap (error correction model). For long-term forecasts, both techniques are outperformed by the fundamental random walk with drift model. They beat the latter model for short and medium-term forecasts, but are usually outperformed by (or are close to) the random walk model with drift.

[25] The only exception is Japan, for which the random walk and the AR(1) forecasts were slightly better than the random walk with drift for both the observed and the fundamental price. However, using them as the benchmark does not change the results presented in the next sections.

[26] Using the root mean square error to measure the forecast accuracy does not change the results.

[27] Note that the models with and without breaks in inflation are almost similar for the United Kingdom.

Fig. 8. Forecast accuracy of fundamental models



Each panel compares the forecast accuracy of different models with the forecast accuracy of the random walk with drift model (benchmark). The black line corresponds to the fundamental model without break, the doted line to the model with break in inflation and the dashed line to the model with money illusion. Each panel gives the ratio of the model's average absolute errors over the benchmark's average absolute error. A value under 1 indicates that the model is more accurate than the benchmark. The comparison is made for different forecasting horizon (in months, horizontal axis).

United States and the United Kingdom, with a maximum improvement of about 15% for an 6-year forecast horizon. For Switzerland, the fundamental model *with breaks* is more accurate than the benchmark for horizons longer than 4 years. The improvement is of 20% after 6 years.[28]

These results tend to confirm the presence of a relatively strong link between the fundamental and the observed stock prices in all countries. However, the most accurate model differs from one country to the other. The model with money illusion is the best in the United States (for horizons longer than 2 years) and in the United Kingdom (for horizons between 8 months and 3 years). The model with breaks in inflation is the most accurate in the United Kingdom (for horizons longer than 3 years), in Japan (for horizons longer than 5 years) and in Switzerland (for horizon longers than 4 years).

[28] Lioui and Rangvid (2007) also find that PD ratios combined with inflation level give better forecasts than simple PD ratios for short horizons.

8 Is it really worth computing the CAPM fundamental price?

The goal of this section is to establish whether the efforts made in constructing the C-CAPM fundamental price are really worthwhile or whether computationally less intensive fundamental models give similar results. For this, I first decompose the fundamental price to see which component is of significant importance to the fundamental price. Secondly, I compare the forecasts of simpler models to see if they outperform the C-CAPM in terms of forecast accuracy. Finally, I analyze the fundamental forecasts at different points in time to see if the C-CAPM fundamental price is more relevant in certain periods than in others.

8.1 Fundamental price decomposition

In this section, I decompose the fundamental price into the main innovations that have been added by this paper to the original method of Campbell and Shiller (1988b). The goal is to see which components really change the results. For that, I first estimate a fundamental price without money illusion, without breaks, without second moments and over all samples, as in Campbell and Shiller (1988b). Secondly, I estimate the same model but by using, at each point in time, only the information that was available at that time (out-of-sample estimation). Thirdly, I add the estimated conditional second moments. Finally, I introduce breaks in inflation and money illusion. Table 3 presents the fraction of the absolute deviation from the mean which is explained by the different components. [29]

The first observation is that, in each country, the in-sample estimated fundamental PD ratio without money illusion does not differ much from the average PD ratio (i.e. it is almost a flat line). With the possible exception of Switzerland, introducing out-of-sample estimation does not make the fit of the model much better. Second moments do not help to make the fit of the model better either (except for Japan). Introducing breaks in inflation significantly increases the fit of the model for the United States and Switzerland. Finally, introducing money illusion significantly increases the fit for all countries but Switzerland. Note, however, that having a better fit does not necessarily imply better forecasts. In Japan, for example, the model with money illusion has a better fit

[29] This coefficient is computed by $1 - \sum |pd_t - pd_t^*| / \sum |pd_t - \overline{pd}_t|$ where pd_t is the observed log PD ratio, $\overline{pd}_t$ its average and pd_t^* the estimated fundamental PD ratio. This coefficient is a measure of the fit of a model. A coefficent equal to 0% means that the model has the same fit as the mean. A coefficient of 100% means that the model fits the observed dynamic perfectly. This coefficient is similar to the coefficient of determination R^2 but places less weight on extreme observations.

Table 3
Percenage of absolute deviation from the mean explained by the fundamental model

Sample	In	Out	Out	Out	Out
Moments	1st order	1st order	2nd order	2nd order	2nd order
Break	No	No	No	Yes	Yes
Money illusion	No	No	No	No	Yes
United States	0.96%	5.26%	5.27%	25.91%	45.81%
United Kingdom	1.77%	1.82%	1.91%	1.93%	28.45%
Japan	3.94%	4.20%	14.91%	8.78%	31.63%
Switzerland	-0.44%	7.36%	7.36%	42.43%	28.17%

than the one with breaks in inflation but the latter gives better forecasts.

8.2 Forecasts with alternative fundamental models

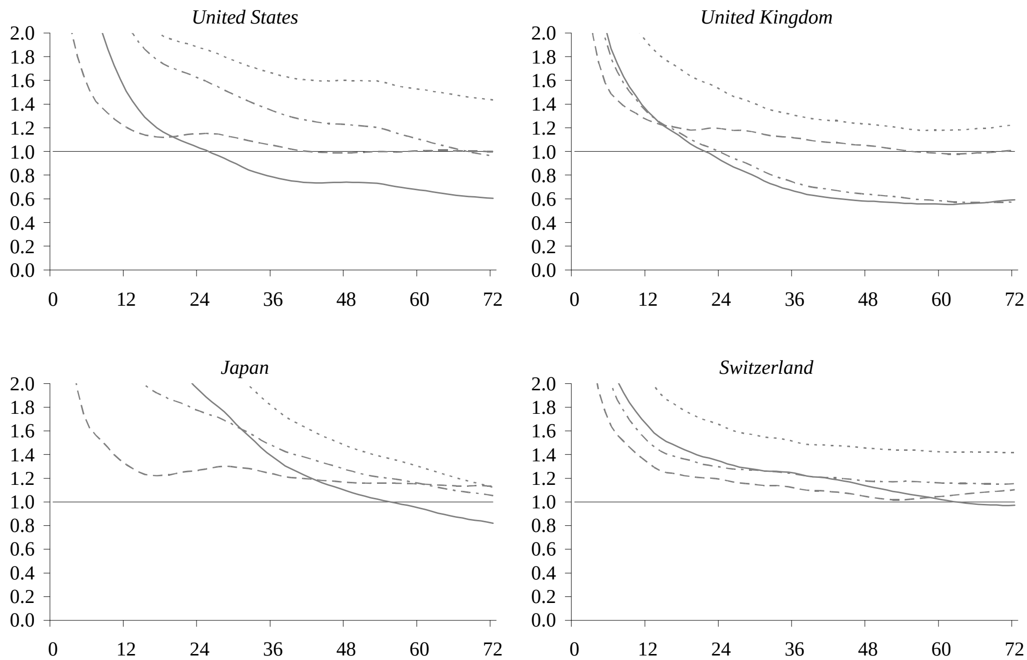
This section compares the forecast accuracy of the C-CAPM fundamental price developed in this paper with simpler ad-hoc fundamental models. I compare the C-CAPM model with three simpler models:

(1) **Trend model**: in this model, the fundamental price is determined by fitting a linear trend with the observed prices.
(2) **Hodrick-Prescott filter model**: in this model the fundamental price is determined by estimating a Hodrick-Prescott filter with the observed price. [30]
(3) **Moving average PD ratio**: in this model, the fundamental price is determined by the 10-year moving average of the PD ratio.

At every period, each fundamental price is estimated with the data available at that time. I chose to compare the forecasts of the alternative fundamental models with the ones of the fundamental model with money illusion for the US and with breaks in inflation for the UK, Japan and Switzerland (cf. section 7.2). The alternative fundamental models correspond to measures that are commonly used by practitioners to estimate imbalances on stock markets. For

[30] The smoothing parameter is set to 230,400, which corresponds to the hypothesis of an average gap of 20% between the fundamental and the observed price and an average annual change of 4% in the long-term stock price trend.

Fig. 9. Forecast accuracy of alternative fundamental models



Each panel compares the forecast accuracy of different fundamental models with the forecast accuracy of the random walk with drift model (benchmark). The black line corresponds to the C-CAPM model, the doted line to the trend model, the dashed line to the Hodrick-Prescott filter model and the dot-dashed line to the moving average PD ratio model. Each panel presents the ratio of the model's average absolute errors over the benchmark's average absolute error. A value under 1 indicates that the model is more accurate than the benchmark. The comparison is made for different forecasting horizon (in months, horizontal axis). The C-CAPM model with money illusion is used for the US. The C-CAPM without break in inflation is used for the UK, Japan and Switzerland.

example, Borio and Lowe (2002) use the Hodrick-Prescott filter to identify bubbles in different stock markets. A constant PD ratio corresponds to the Gordon (1962) model. The moving average PD ratio can thus be considered as a Gordon model based on the recent observed dynamic.

Figure 9 shows the forecast accuracy of the alternative fundamental models. None of the alternative models can simultaneously outperform the benchmark or the C-CAPM fundamental model. This means that when the C-CAPM is more accurate than the benchmark, it is also more accurate than simpler fundamental models. Thus the C-CAPM fundamental model developed in this paper adds some value to simpler ad-hoc fundamental models.

8.3 When are the fundamental forecasts the most accurate?

The previous sections examine the forecast accuracy of C-CAPM for the whole sample. However, it might happen that this model performs better in some periods than in others. This section addresses this question and, more particularly, the question of whether there is a link between the wideness of the

Table 4
Correlation between absolute gaps and forecast errors

Forecast horizon	1 year	2 years	3 years	4 years	5 years
United States	-0.3114*	-0.0080	0.1655*	0.2666*	0.4623*
United Kingdom	-0.1486*	0.1830*	0.4653*	0.5793*	0.6162*
Japan	-0.5719*	-0.1851*	0.0742	0.2845*	0.3582*
Switzerland	-0.2879*	-0.0821	-0.0637	-0.0531	0.1018*

* denotes the rejection of the null hypothesis of a coefficient equal to 0 at a 5% confidence level. The correlation is measured with the Spearman coefficient. A positive value means that a (positive or negative) wider gap implies an increase in the accuracy of the fundamental forecasts in comparison to the forecasts based on the observed price. I use the fundamental model with money illusion for the United States and the one with breaks in inflation for the other countries.

gap between the fundamental and the observed prices and the precision of the forecasts.

Table 4 gives the relation between the wideness of the gap and the forecast error. It presents the Spearman correlation [31] between the absolute value of the gap and the forecast accuracy improvement made by using the fundamental model. [32] A positive correlation implies that a (positive or negative) wider gap increases the accuracy of fundamental forecasts in comparison to forecast based on the observed price.

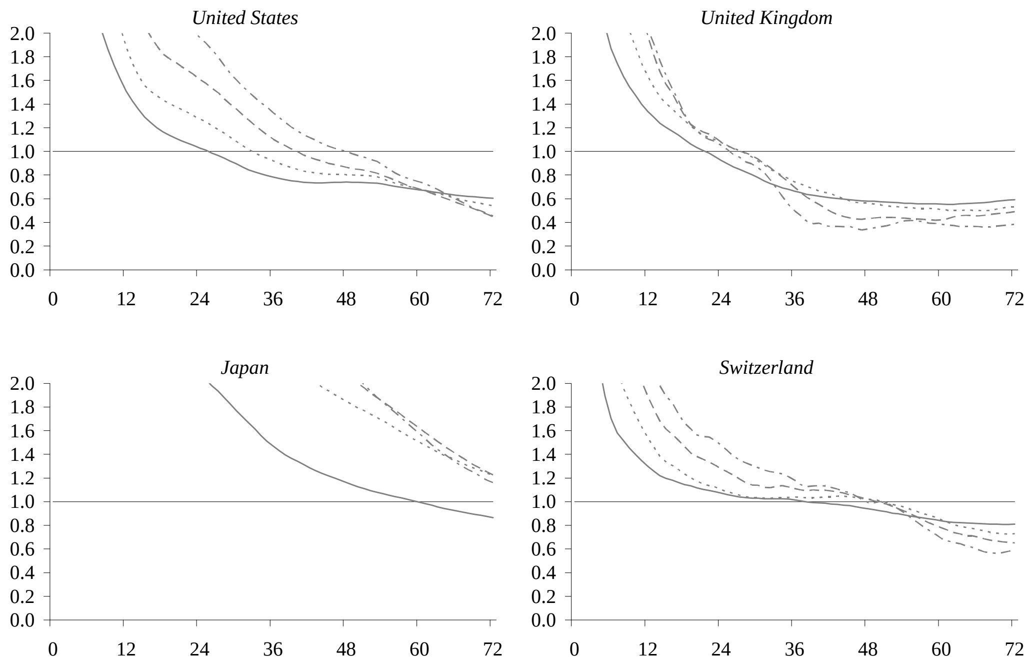
The correlation analysis shows that in all countries, for horizons longer than 3 years in the US, 2 years in the UK, 4 years in Japan and 5 years in Switzerland, a wider gap implies more accurate forecasts for the fundamental model (with money illusion for the US and with breaks in inflation for the others). Since more accurate forecasts are a sign of a stronger link between the fundamental and the observed price, the positive correlation suggests that the further apart the two prices are, the more they tend to return towards each other in the long term. In other terms, the "attraction force" exerted by the fundamental price on the observed price become stronger as the distance between them increases. We can also see that the strength of the link increases with the forecast horizon.

Building on these conclusions, one could adopt the following strategy: if the

[31] Since the form of the function linking the (absolute) gap with the forecast error is a priori not known, I prefer to use the Spearman rank-order correlation coefficient rather than the traditional (linear) correlation (Pearson coefficient). The Spearman correlation coefficient is independent of the form taken by the function. The Spearman rank-order correlation coefficient measures the linear correlation between the ranks of each observation.

[32] This variable takes a positive value when the fundamental price forecasts are better than the observed price forecasts, and a negative value when they are worse.

Fig. 10. Forecast accuracy for periods with large gaps



Each panel compares the forecast accuracy of the random walk with drift model to the forecast accuracy of the fundamental random walk with drift model for forecasts made on the whole sample (grey line), for forecasts made when the (absolute) value of the gap is greater than its median (doted line), than its 25 percentile (dashed line) and than its 10 percentile (dashed-dotted line). Each panel presents the ratio of the model's average absolute errors over the benchmark's average absolute error. A value under 1 indicates that the model is more accurate than the benchmark. The comparison is made for different forecasting horizon (in months, horizontal axis). The C-CAPM model with money illusion is used for the United States. The C-CAPM with breaks in inflation is used for the United Kingdom, Japan and Switzerland.

gap is large, then one should use the fundamental price to make forecasts, otherwise the observed price should be used. Figure 10 compares the forecast accuracy of the fundamental random walk with drift for any value of the gap with those made when the gap is large. It shows that, for horizons longer than 5 years, the forecast accuracy improves by 15-20 percentage points in the United States, United Kingdom and Switzerland, if the forecasts are made when the gap is large. In the United Kingdom, this strategy improves any forecasts with a horizon longer than 3 years. I find no improvement for Japan.

9 Conclusion

The results presented in this paper can be interpreted and exploited at different levels. The most obvious practical result is the ability of the C-CAPM fundamental model to forecast future prices for short horizon. As mentioned in the paper, the predictability of stock prices is not new. Several studies have shown that simple ratios such as dividend-price ratios or price-earnings ratios are useful for forecasting stock prices in the long term. Typically, these ratios are able to forecast price for horizons longer than five years. What is new here is that the C-CAPM fundamental model significantly shortens the horizons for which the forecast accuracy can be improved. Of course, the improvement depends on the market and on the period in which the forecast is made, but the results presented in this paper suggest that it is generally possible to find a way to combine the observed and the fundamental prices to improve forecasts for horizons shorter than five years. While long-term forecasts are of little interest for traders or portfolio managers, whose performance is evaluated on shorter intervals, forecast horizons in the range of those proposed in this paper might find their place in longer term portfolio allocation strategies. With that in mind, it would be useful to study in greater depth how significant and reliable the improvement made by the fundamental model is and whether a portfolio based on fundamental forecasts would have generated excess returns in the past.

The C-CAPM fundamental model is also of interest for central banks and international organizations such as the IMF or the BIS. Indeed, such institutions show a growing interest in identifying imbalances on asset markets. Their fear is that such imbalances will eventually unwind and that their correction might have a significant impact on the economy. Thus, by detecting imbalances early enough, central banks hope to identify factors that could help them in predicting the evolution of the economy and choosing an appropriate policy. Two results indicate that the imbalances measured by the C-CAPM fundamental model would be relevant in this framework: firstly, there is strong empirical evidence of a link between the C-CAPM fundamental price and the observed stock price and secondly, the forecast horizon for which the fundamental model is helpful corresponds to the horizon, which is normally considered as pertinent for central bank policy (up to 3 or 5 years). In this context, an improvement of the model would be to relax assumption 5 and extend the set of variables used to forecast the fundamentals. As mentioned by Campbell (1999), consumption is not well forecasted by its own history. It is therefore unlikely that agents will rely purely on its dynamic to form their expectations. Expanding the information set to other variables would refine the estimation of the gap between the price and its fundamental value and give a better measure of potential imbalances.

Central bankers may also be interested by the signs of money illusion found for the United States, and to a lesser extent for the United Kingdom. In the presence of money illusion, inflation induces mispricing on stock markets. Therefore, as suggested by Campbell and Vuolteenaho (2004), a by-product of inflation-stabilizing monetary policies is to reduce the volatility of mispricing and foster stock market efficiency. More generally, the results of this paper demonstrate the importance of inflation in stock markets. The decrease in inflation observed in the last 20 years seems to be a plausible explanation for the increase in PD ratios, whether through its impact on real fundamentals or on nominal ones via money illusion phenomenons.

Finally, from an academic point of view, the results presented in this paper seem to rehabilitate to some extent the empirical relevance of the C-CAPM with power utility. Indeed, since the seminal paper of Mehra and Prescott (1985) documenting the equity premium puzzle raised by using the C-CAPM with power utility, this model has been the target of multiple critiques. Numerous articles have contested or defended its use in the context of stock prices. In recent years, the trend seems to have been moving toward modified utility functions such as habit formation or loss aversion. Most of these studies are based on the observation of stock returns. This paper sheds new light on the C-CAPM with power utility by looking at stock prices instead of returns. From this point of view, the empirical evidence is kinder with the power utility C-CAPM, at least for the long-term evolution of stock prices. Note that the methodology developed in this paper can be easily extended to any other linear SDF model. It would be interesting to establish whether utility functions that perform better than the power utility for stock returns also give better results for the long-term dynamic of stock prices.

A Proofs

A.1 Second-order Taylor expansion of the log PD ratio

We start from the definition of the fundamental PD ratio in equation (5)

$$PD_t = E_t \sum_{i=1}^{\infty} \prod_{j=1}^{i} M_{t+j} \gamma_{t+j} = E_t \left(PD_t^* \right) = \sum^{s \in S} \lambda_s PD_{s,t}^* \tag{A.1}$$

where $S = \{s_1, \ldots, s_N\}$ is the set of every possible state of nature s and λ_s is the probability to observe the state of nature s. Taking the log of this equation yields

$$\ln E_t \left(PD_t^* \right) = \ln \sum^{S} \lambda_s PD_{s,t}^* = \ln \sum^{S} \lambda_s e^{pd_{s,t}^*} = f\left(pd_{1,t}^*, \ldots, pd_{N,t}^* \right) \tag{A.2}$$

The second-order Taylor expansion of $f\left(pd_{1,t}^*, \ldots, pd_{N,t}^* \right)$ around $\overline{pd}_t^*$ is

$$f\left(\overline{pd}^*_t,\ldots,\overline{pd}^*_t\right)+\sum^{s\in S}\left.\frac{\partial f}{\partial pd^*_{s,t}}\right|_{pd^*_{s,t}=\overline{pd}^*_t}\left(pd^*_{s,t}-\overline{pd}^*_t\right)+$$
$$+\frac{1}{2}\sum^{s_1\in S}\sum^{s_2\in S}\left.\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}\right|_{pd^*_{s,t}=\overline{pd}^*_t}\left(pd^*_{s_1,t}-\overline{pd}^*_t\right)\left(pd^*_{s_2,t}-\overline{pd}^*_t\right)+ \tag{A.3}$$
$$+R_t$$

where $\overline{pd}^*_t=\sum^S \lambda_s e^{\overline{pd}^*_t}$ and R_t is the remainder of the Taylor expansion.

We have that

$$f\left(\overline{pd}^*_t,\ldots,\overline{pd}^*_t\right)=\ln\sum^S \lambda_s e^{\overline{pd}^*_t}=\overline{pd}^*_t \tag{A.4}$$

The first-order term is

$$\sum^{s\in S}\left.\frac{\partial f}{\partial pd^*_{s,t}}\right|_{pd^*_{s,t}=\overline{pd}^*_t}\left(pd^*_{s,t}-\overline{pd}^*_t\right)=$$
$$=\sum^{s\in S}\left.\frac{\lambda_s e^{\overline{pd}^*_t}}{\sum^S \lambda_s e^{\overline{pd}^*_t}}\right|_{pd^*_{s,t}=\overline{pd}^*_t}\left(pd^*_{s,t}-\overline{pd}^*_t\right)=$$
$$=\sum^{s\in S}\lambda_s\left(pd^*_{s,t}-\overline{pd}^*_t\right) \tag{A.5}$$

The second derivative in the second-order term is

$$\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}=-\frac{\lambda_{s_1}e^{pd^*_{s_1,t}}\lambda_{s_2}e^{pd^*_{s_2,t}}}{\left(\sum^S \lambda_s e^{pd^*_{s,t}}\right)^2}\text{ if } s_1\neq s_2 \tag{A.6}$$
$$\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}=\frac{\lambda_{s_1}e^{pd^*_{s_1,t}}\sum^S \lambda_s e^{pd^*_{s,t}}-\left(\lambda_{s_1}e^{pd^*_{s_1,t}}\right)^2}{\left(\sum^S \lambda_s e^{pd^*_{s,t}}\right)^2}\text{ if } s_1=s_2 \tag{A.7}$$

Evaluated at $\overline{pd}^*_t$, the second derivative is

$$\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}=-\lambda_{s_1}\lambda_{s_2}\text{ if } s_1\neq s_2 \tag{A.8}$$
$$\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}=\lambda_{s_1}\left(1-\lambda_{s_1}\right)\text{ if } s_1=s_2 \tag{A.9}$$

Given that, the second-order term is

$$\sum^{s_1\in S}\sum^{s_2\in S}\left.\frac{\partial^2 f}{\partial pd^*_{s_1,t}\partial pd^*_{s_2,t}}\right|_{pd^*_{s,t}=\overline{pd}^*_t}\left(pd^*_{s_1,t}-\overline{pd}^*_t\right)\left(pd^*_{s_2,t}-\overline{pd}^*_t\right)=$$

$$= \sum^{s \in S} \lambda_{s_1} \left(pd^*_{s,t} - \overline{pd}^*_t \right)^2 \tag{A.10}$$

Recollecting all these results, we find that the Taylor expansion is

$$f\left(pd^*_{1,t}, \ldots, pd^*_{N,t}\right) = \overline{pd}^*_t + \sum^{s \in S} \lambda_s \left(pd^*_{s,t} - \overline{pd}^*_t \right) +$$

$$+ \frac{1}{2} \sum^{s \in S} \lambda_{s_1} \left(pd^*_{s,t} - \overline{pd}^*_t \right)^2 + R_t \tag{A.11}$$

and thus

$$\ln E_t \left(PD^*_t \right) = E_t \left(pd^*_t \right) + \frac{1}{2} V_t \left(pd^*_t \right) + R_t \tag{A.12}$$

A.2 First-order approximation of the logarithm of a sum

Campbell, Lo, and MacKinlay (1997) show that it is possible to approximate the logarithm of a sum by the sum of logarithms. First consider

$$\ln \left(1 + PD^* \right) = \ln \left(1 + e^{pd^*} \right) \tag{A.13}$$

where $pd^* = \ln PD^*$. The second part of this equation can be approximated by a standard Taylor approximation around its mean. Define $f(x) = \ln \left(1 + e^{pd^*} \right)$ and take the Taylor approximation of it around its mean:

$$f\left(pd^*\right) \approx f\left(\overline{pd}^*\right) + f'\left(\overline{pd}^*\right)\left(pd^* - \overline{pd}^*\right) \tag{A.14}$$

with $f'\left(\overline{pd}^*\right) = e^{-\overline{pd}^*} / \left(1 + e^{-\overline{pd}^*}\right)$. Define $\rho = 1/\left(1 + e^{-\overline{pd}^*}\right)$ and plug it into the previous equation to get the final result

$$\ln \left(1 + PD^* \right) \simeq \kappa + \rho pd^* \tag{A.15}$$

with $\kappa = -\ln \rho - (1 - \rho) \ln (1/\rho - 1)$.

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Part II

Time Variations in the Equity Premium: Is it Habit Formation or Loss Aversion? ⋆

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Abstract

The large spread between equity returns and risk-free rates (the "equity premium puzzle") has been the subject of intense debate. Two main families of models claim to solve this puzzle: habit-formation models and loss-aversion models. The goal of this paper is to assess empirically which of them best fits the observed excess returns. In order to do this, I first show how to express both models as linear stochastic discount factor models. This gives explicit and testable constraints for the excess return variations. I then compare the theoretical evolution generated by these models with the observed path. I find that the constant relative risk aversion model and the external habit model are not likely to correspond to the observed data. The loss aversion model and the internal habit model could fit the observed excess return evolution.

Key words: Excess stock return; Habit formation; Loss aversion.

JEL classification: E21, E44, G12.

⋆ The opinions expressed herein are those of the author and do not necessarily reflect the views of the Swiss National Bank.

1 Introduction

In 1985, Mehra and Prescott introduced the academic and financial profession to the "equity premium puzzle" (Mehra and Prescott, 1985). They claimed that the difference between returns on equities and risk free returns is too large to be compatible with a traditional asset pricing model, which uses a representative agent with constant relative risk aversion. They argued that the coefficient of relative risk aversion (CRRA) estimated with such a model is much higher than the value found in experimental studies. This finding has been replicated several times by other studies. Campbell (1999), for example, estimates that the CRRA must be equal to 316 to match the equity premium observed in the US between 1947 and 1996. Experiments, however, show that the average CRRA usually lies between 0 and 5, with 10 being considered as an extreme value.

Since then, numerous solutions have been proposed to reconcile the observed equity premium with an asset pricing model based on a utility-maximizer agent. It is beyond the scope of this paper to survey the literature still being generated by Mehra and Prescott's article.[1] The goal of this paper is rather to propose a new way to test two categories of models that have recently caught up much of researchers' attention on this topic: *habit formation models* (Abel, 1990, Constantinides, 1990, Campbell and Cochrane, 1999) and *loss aversion models* (Benartzi and Thaler, 1995, Barberis, Huang, and Santos, 2001, De Giorgi, Hens, and Mayer, 2007). Both models claim to explain the equity premium puzzle by using alternative utility functions. Habit formation models postulate that the utility of the representative agent should be a function of the difference between her consumption and a habit, rather than of consumption only. The habit represents a minimum level under which consumption cannot fall (i.e. a subsistence level). Loss aversion utility functions have two components: the traditional "consumption utility" based on consumption and the "gain-loss utility" which is based on the deviation of consumption from a reference level. Thus a loss-averse agent has an additional source of utility: she gets (loses) some "extra" utility if she consumes more (less) than her reference level. The particularity of loss aversion models is that losses are more "painful" than gains and thus a loss generates a greater decrease in utility than the increase in utility that would be derived from a gain of the same size. The early papers using these utility functions provide theoretical explanations for the equity premium puzzle as well as empirical calibrations which back their conclusions.

This paper contributes to the literature on the equity premium in two ways:

[1] See e.g. Mehra and Prescott (2003) for a survey of selected issues on the equity premium puzzle.

firstly, it shows how to express both habit formation models and loss aversion models in a common framework, namely as linear stochastic discount factor (SDF) asset pricing models. The SDF form has two advantages: first, it provides two different sets of *explicit linear constraints on the dynamic of excess returns*, one for habit formation models and one for loss aversion models. The second advantage is that the linear SDF form can be easily estimated using a vector autoregressive (VAR) model with multivariate GARCH errors.

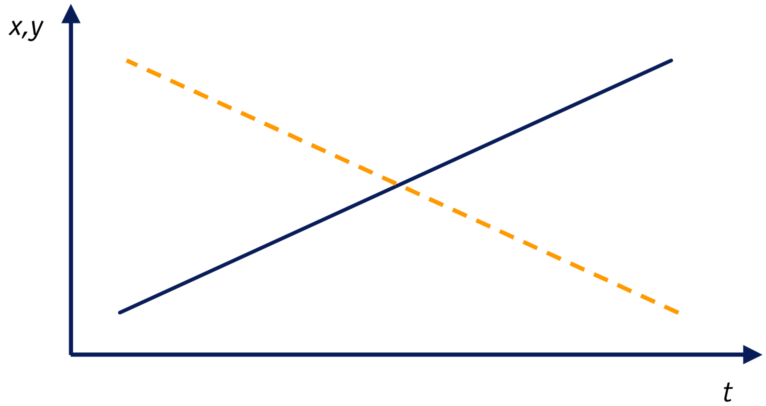
The combination of both advantages of the linear SDF form is at the roots of the second innovation of this paper: by comparing the empirical variations in excess returns with the theoretical constraints of habit formation models or those of loss aversion models, it is possible to check whether the theoretical dynamic of one model corresponds to the observed variations or if it substantially deviates from it. It is even possible to statistically test, e.g. with traditional Wald test, if the null hypothesis of one model is rejected at a significant confidence level. The ultimate goal of this exercise is to determine which model (if any) is the closest to the observed excess returns. [2]

Several previous studies have tried to assess the empirical validity of habit formation or loss aversion models. The usual way to do this is to use GMM estimation to match the model with a set of observed unconditional moments and then to check whether the estimated coefficients are plausible or if the estimated model generates moments for other variables that are close to reality. These studies provide contradictory evidence for habit formation models. On the positive side, GMM estimations generally give parameters in line with theory (cf. e.g. Hyde and Sherif (2005) for UK data, Allais, Cadiou, and Dées (2000) for G7 countries and Menzly, Santos, and Veronesi (2001) for cross-sectional stock returns). Li (2001) and Li and Zhong (2005) show that habit formation models can explain some part of excess return predictability and outperform power utility models in forecasting excess returns. Other studies are less positive. Tallarini and Zhang (2005) reject a habit formation model with the Efficient Method of Moments at a 1% level (but not at a 0.1% level) and Chapman (2002) cannot reproduce Constantinides (1990) results with post-war data. More doubts are cast on habit formation models when their dynamic properties are examined. Li (2001) finds that volatility of returns is positively related to surplus consumption, which is in contradiction with Campbell and Cochrane (1999) model. Duffee (2005) estimates that the price of risk generated by habit formation does not vary enough to justify the evolution of the equity premium. Allais, Cadiou, and Dées (2000), Lettau and Uhlig (2000) and Pijoan-Mas (2007) show that the consumption implied by habit formation models is much smoother than what is observed in reality. [3] Let-

[2] Note that the methodology presented here can be used to test the validity of any model expressed in the form of a linear SDF model.

[3] Jermann (1998) and Budría (2004) overcome this problem by introducing in-

Fig. 1. Unconditional vs. conditional moments



tau and Uhlig (2000) and Otrok, Ravikumar, and Whiteman (2002) find that habit formation models induce counterfactual cyclical behaviour (on labour markets or concerning the evolution of the consumption variability and the equity premium, respectively). On the other hand, Woehrmann, Semmler, and Lettau (2005) find that habit formation models generate a path for Sharpe ratios that better fits the observed path than a model with CRRA does. Empirical evidence on loss aversion models is scarcer. Using GMM estimation, Yogo (2008) finds parameters corresponding to the theory and smaller pricing errors for a model incorporating loss aversion.

The methodology presented in this paper completes and extends the previous studies in three directions. Firstly, it provides a formal test of the coefficients of the excess returns dynamic. This make it possible to compare the entire path of the different models rather than just their first or second moments. Comparing the observed and the theoretical paths rather than unconditional moments is methodologically more robust. Indeed, one can easily imagine a model that has the same first and second moments as its theoretical counterpart but with a totally different evolution (e.g. see Figure 1). Secondly, the approach used here is based on conditional moments rather than unconditional ones. This is closer to reality because investors act according to the information available at the time when their decision is made (conditional information set). Finally, this paper compares habit formation and loss aversion models in the same methodological framework, which makes the comparison (and the potential ranking of models) easier and coherent. To my knowledge, it is the first study that simultaneously compares the performance of habit formation models and loss aversion models.

Specifically, I test two habit formation models (the internal habit model of Constantinides, 1990, and the external habit model of Campbell and Cochrane, 1999) and two loss aversion models (one in which the reference level is based

vestment costs and Boldrin, Christiano, and Fisher (2001) by assuming inertia in production technology adjustments.

on expectations, as in Benartzi and Thaler, 1995, and Barberis *et al.*, 2001, and one in which the reference level is based on past consumption, as in Yogo, 2008). I use monthly US data on consumption and stock market excess returns over the period 1959-2005. The loss aversion models tested here differ from their original specification, as the losses and gains are expressed in terms of consumption rather than wealth.

I find that the constant relative risk aversion model and the external habit model are strongly rejected by the data. The observed excess return evolution do not correspond to the path generated by these models. The constraints that they impose on the dynamic are formally rejected by both Wald and Likelihood ratio tests. The validity of the loss aversion model using past consumption as the reference level is also doubtful. Although its restrictions cannot be rejected at a 1% confidence level, the values of the estimated parameters cast some doubt on it. The two remaining models (internal habit formation and loss aversion with expected consumption as the reference point) are possible candidates for explaining the excess returns evolution. They cannot both be rejected at a 1% confidence level and have plausible parameters. The loss aversion model has a slight advantage over the habit formation model as the Wald test does not reject its constraints at a 10% level. [4]

In addition to the ranking of the different models, I find two interesting facts concerning the evolution of excess returns. Firstly, as already documented by Duffee (2005), the conditional covariance between excess returns and consumption growth seems to be negatively related to excess returns. This is in contradiction to the constant relative risk aversion model and the external habit model. Secondly, consumption growth variance seems to play a role in the excess return evolution. This factor does not appear in any of the proposed models and thus might constitute a starting point for the development of a more adequate model.

Section 2 presents the asset pricing equation used to test the model as well as the test method. Section 3 shows how to express habit formation and loss aversion models in the form of a linear SDF model. Section 4 gives the empirical results. Section 5 concludes.

2 Methodology

The methodology used in this paper is based on the tests for exact linear rational expectations models developed by Hansen and Sargent (1981). This

[4] Both Wald and Likelihodd ratio test reject the internal habit model at a 5% confidence level.

method follows three steps:

(1) Draw up the theoretical model in order to derive some hypotheses (or restrictions) on the joint dynamic of the variables.
(2) Estimate the empirical joint dynamic of the variables.
(3) Test whether the estimated dynamic is compatible with the theoretical hypothesis made in the first step.

This section shows how to derive hypotheses on the joint dynamic from any linear SDF model. Section 3 then explains how to express the traditional CRRA model, the habit formation model and the loss aversion model as linear SDF models.

2.1 The C-CAPM no-arbitrage equation

The hypotheses on the joint dynamic are directly derived from the no-arbitrage equation

$$E_t\left(M_{t+1}R_{t+1}\right) = 0 \tag{1}$$

where M_{t+1} is the SDF, or pricing kernel, and R_{t+1} is the excess return of a risky asset over a risk-free bond. This no-arbitrage equation simply states that the *discounted* expected excess return of risky assets is equal to zero or, in other terms, that *ex-ante* there is no systematical discounted excess return.

The SDF M_{t+1} can take different forms, depending on the model used to describe the behaviour of asset prices. Each model considered in this paper has a different SDF, but they share a common feature: they are derived from the Consumption-based Capital Asset Pricing Model (C-CAPM). They all start with the same maximisation problem, in which a representative agent allocates her resources between savings and consumption in order to maximise the utility of her present and future consumption. The SDF of the C-CAPM is

$$M_{t+1} = \beta\frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)} \tag{2}$$

where β is a subjective discount factor and $U'\left(C_t\right)$ is the marginal utility of consumption at time t (see Cochrane, 2001, for more details). The only difference between the models presented here concerns the form taken by the representative consumer utility function $U\left(C_t\right)$. The SDF associated to the different utility functions are presented in Section 3. In the next section, I will use the general SDF model in equation (1) to explain the methodology and then, when presenting each utility function, I will point out the specific characteristics of each SDF.

Following Söderlind (2006), if we assume that log SDF m_{t+1} is normally distributed and that m_{t+1} and the excess return R_{t+1} have a bivariate mixture normal (conditional) distribution, we have that

$$E_t\left(R_{t+1}\right) = -Cov_t\left(m_{t+1}, R_{t+1}\right) \tag{3}$$

Economically, this equation specifies that the *expected* excess return is equal to the *expected* risk premium for the risky asset. This no-arbitrage condition constitutes the model tested in this paper. The test consists of verifying whether the empirical excess return evolution is compatible with equation (3).

Note that traditionally, in the literature, another linearisation of equation (1) is used by assuming log normal returns. Under this assumption, a correction for Jensen's effect is added to the no-arbitrage equation (3), which becomes $E_t\left(R_t\right) + V_t\left(R_{t+1}\right) = -Cov_t\left(m_{t+1}, R_{t+1}\right)$. Empirically, asset returns distributions are well known to have fat tails. The mixture of normal distribution used here is more flexible and can take different shapes that allow for such fat tails. It is therefore more likely to fit the returns real distribution than the log normal distribution. The cost of this greater flexibility for the returns distribution is to assume that the (conditional) log SDF is normally distributed. Söderlind (2006) shows that for SDF based on consumption, this hypothesis is much more likely to be true than for excess returns.

2.2 Specifying the joint dynamic of the variables

Equation (3) suggests that second moments of the joint distribution of m_{t+1} and R_{t+1} are time varying and that the evolution of m_{t+1} and R_{t+1} is a function of their second moments. These characteristics corresponds to the VAR-GARCH-M model (Pelloni and Polasek, 1999, Polasek and Ren, 2000). This model is the multivariate version of the univariate GARCH-in-mean model of Bollerslev, Engle, and Wooldridge (1988). The VAR-GARCH-M model takes the following form

$$y_t = \mu + Ay_{t-1} + Bh_{t-1} + De_{t-1} + \varepsilon_t \tag{4}$$

$$\begin{aligned} h_t &= \omega + Fh_{t-1} + Ge_{t-1} \\ h_t &= vech\left\{H_t\right\} \\ e_t &= vech\left\{\varepsilon_t\varepsilon_t'\right\} \\ H_t &= E_{t-1}\left(\varepsilon_t\varepsilon_t'\right) \end{aligned} \tag{5}$$

where $y_t = \left[R_t, \ldots, R_{t-j}, m_t, \ldots, m_{t-j}\right]$ is a vector of observations, A, B, C and D are matrices of coefficients, μ and ω are vectors of coefficients and ε_t is a vector of error terms. The *vech* operator converts the lower triangle of a symmetric matrix into a vector. Finally, H_t is the expected covariance[5] matrix at time t, The model is made up of three parts: equation (5) shows that a) the covariances follow a GARCH model and equation (4) shows that b) the variables are a linear function of their past values (as in a VAR model) and c) of the conditional second moments (as in a traditional GARCH-M model).

2.3 The joint dynamic implied by the C-CAPM

The next step is to derive the restrictions that the no-arbitrage equation (3) imposes on the coefficient of the VAR-GARCH-M model. Let us define $\mathbf{y}_t = \begin{bmatrix} 1 & y_t & h_t & e_t \end{bmatrix}$ and rewrite the VAR-GARCH-M with this new variable

$$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t \tag{6}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mu & A & B & D \\ \omega & 0 & F & G \\ \omega & 0 & F & G \end{bmatrix} \text{ and } \boldsymbol{\varepsilon}_t = \begin{bmatrix} 0 \\ \varepsilon_t \\ 0 \\ \upsilon_t \end{bmatrix}$$

where υ_t is a vector of white noise. Rewriting equation (3) in terms of the vector $\mathbf{y}_t$ gives

$$\mathbf{g}' E_t\left(\mathbf{y}_{t+1}\right) = 0 \tag{7}$$

where $\mathbf{g}'$ is a vector that selects the variables appearing in equation (3); namely $E_t R_{t+1}$ and $E_t h_{t+1}^{12} = Cov_t\left(R_{t+1}, m_{t+1}\right)$.

From equation (6), we obtain the following expectation about y_{t+1}

$$E_t\left(\mathbf{y}_{t+1}\right) = \mathbf{A}\mathbf{y}_t \tag{8}$$

Plugging this expectation into equation (7) gives

$$\mathbf{g}'\mathbf{A}\mathbf{y}_t = 0 \tag{9}$$

Thus, the coefficients $\mathbf{A}$ of the variables' dynamic must satisfy the constraint

$$\mathbf{g}'\mathbf{A} = 0 \tag{10}$$

in order for equation (3) to hold. The vector $\mathbf{g}'$ depends on the form of the SDF, which differs for each model studied in this paper. For the general SDF model, this constraint implies that the excess return evolution should be a

[5] In this paper, the term covariance includes also the variances of the variables.

negative function of the *expected* covariance between the excess return and the SDF:

$$R_t = -Cov_{t-1}(R_t, m_t) + \varepsilon_t^R \tag{11}$$

where ε_t^R is a white noise.

The econometric test consists of checking whether the estimated unconstrained VAR with coefficient matrix $\mathbf{A}$ is significantly different from the estimated VAR, where the above constraints are imposed. This can be done using classical Wald tests or Likelihood Ratio tests (see Greene, 2000, p. 150).

3 The different utility functions and their SDF

3.1 Constant relative risk aversion

Power utility, which implies constant relative risk aversion, is the original utility function used by Mehra and Prescott (1985). I use it as a benchmark for habit formation and loss aversion models and to check that the equity premium puzzle is present when conditional moments are used on the specific data set investigated in this paper.

The power utility function is

$$U(C) = \sum_{i=0}^{\infty} \beta^i \frac{C_{t+i}^{1-\sigma}}{1-\sigma} \tag{12}$$

where C_t is the consumption of the representative agent at time t and σ is her coefficient of relative risk aversion. The SDF derived from power utility is

$$m_{t+1} = -\sigma \Delta c_{t+1} \tag{13}$$

where $\Delta c_{t+1} = \ln C_{t+1} - \ln C_t$ is the consumption growth rate. With this SDF the no-arbitrage equation (3) is

$$E_t(R_{t+1}) = \sigma cov_t(\Delta c_{t+1}, R_{t+1}) \tag{14}$$

3.2 Habit formation models

The particularity of habit formation models is that utility is a function of the *gap* between consumption and a habit H_t, instead of consumption level only. The habit is defined as a *subsistence level* under which consumption cannot fall. The habit can be a function of past consumptions ("Catching up with the Joneses" functions introduced by Abel, 1990) or of past and present consumption ("Keeping up with the Joneses" functions introduced by Gali, 1994). There are two main families of habit formation utility functions. In the first one, utility is a function of the difference between consumption and the habit (e.g. Constantinides, 1990, Campbell and Cochrane, 1999). In the second one, utility is a function of the ratio between consumption and the habit (e.g. Abel, 1990, 1999, Gali, 1994). This paper focuses on the first kind of habit formation utility functions, which has recently attracted most of the attention in the literature. The utility function is defined as

$$U(C) = \sum_{i=0}^{\infty} \beta^i \frac{(C_{t+i} - H_{t+i})^{1-\gamma}}{1-\gamma} \tag{15}$$

where γ is a parameter such that $0 < \gamma \leq 1$.

Meyer and Meyer (2005) demonstrate that habit formation utility functions are equivalent to utility functions with decreasing relative risk aversion. Thus, they imply time varying relative risk aversion. If consumption is close to (far from) the habit, the agent has high (low) relative risk aversion.

Habit formation models make a distinction between internal and external habits. The representative agent has internal habits when she takes into account the impact of her present consumption on her current and future habit level. Internal habit models have been introduced by Constantinides (1990) and Abel (1990). External habits (Campbell and Cochrane, 1999) correspond to the case in which agents ignore the impact of their present consumption on habit levels. Chen and Ludvigson (2004) try to determine which alternative best corresponds to the observed data. They conclude their data are better described by internal habit formation than by external habit formation models. Both alternatives are tested in this paper.

3.2.1 Internal habit model

The particularity of internal habit utility model is that agents take into account the impact of their present consumption on present and future habits. Thus, using the utility function in equation (15), the marginal utility of present consumption is equal to

$$U'(C_t) = (C_t - H_t)^{-\gamma} - \sum_{i=0}^{\infty} \beta^i (C_{t+i} - H_{t+i})^{-\gamma} \frac{\partial H_{t+i}}{\partial C_t} \tag{16}$$

The first part of the right hand side of the equation is the utility obtained by consuming one additional unit of a good today. The second part is the reduction of present and future utility that this additional unit induces through its effect on the habit. In other terms, consuming more today brings more utility to the agent today but, at the same time, increases her (present and future) habit, which reduces her (present and future) utility.

As in the original model of Constantinides (1990), I model the habit as a function of past and present consumption. More particularly, the habit is equal to

$$H_t = C_t^{\theta_0} C_{t-1}^{\theta_1} ... C_{t-k}^{\theta_k} \tag{17}$$

where the θ parameters are smaller than one and chosen such that consumption does not fall under the habit.

Appendix B shows how to linearise equation (16) to get the log SDF of the internal habit utility function. This log SDF takes the form:

$$m_{t+1} = \sum_{i=-k+1}^{k+1} \alpha_i \Delta c_{t+i} \tag{18}$$

where α_i are coefficient that mix utility function parameters and linearization parameters.

With internal habits, the SDF is a function of future and past consumption growth rates, whereas it was a function of only the next period growth rate with the simple power utility model. Note also that the coefficients α are different from the coefficient of relative risk aversion.

Using this SDF in equation (3) yields

$$E_t\left(R_{t+1}\right) = -\sum_{i=1}^{k} \alpha_i Cov_t\left(\Delta c_{t+i}, R_{t+1}\right) \tag{19}$$

Note that the conditional covariance between Δc_{t+i} and R_{t+1} is equal to the covariance between the errors made at time t about Δc_{t+i} and R_{t+1} respectively. According to the VAR in equation (6), the error made at time t about the consumption growth rate at time $t+i$ is a linear function of all one-period errors between time t and $t+i$. Thus we have

$$\varepsilon_{t+1}^{R} = R_{t+1} - E_t\left(R_{t+1}\right) = \mathbf{g}_R' \boldsymbol{\varepsilon}_{t+1} \tag{20}$$

$$\varepsilon_{t,t+i}^{\Delta c} = \Delta c_{t+i} - E_t\left(\Delta c_{t+i}\right) = \mathbf{g}_{\Delta c}' \sum_{j=1}^{i} \mathbf{A}^{i-j} \boldsymbol{\varepsilon}_{t+j} \tag{21}$$

where $\mathbf{g}_R'$ and $\mathbf{g}_{\Delta c}'$ are row vectors with zero in each column, with the exception of one 1 in the column corresponding to R_t and Δc_t in the vector y_t, respectively. These vectors select the one period error on R_t and Δc_t respectively. With this notation, and given the fact that the error terms are not autocorrelated, we have that

$$Cov_t\left(\Delta c_{t+i}, R_{t+1}\right) = \mathbf{g}_R' E_t\left(\boldsymbol{\varepsilon}_{t+1}'\boldsymbol{\varepsilon}_{t+1}\right)\left(\mathbf{A}'\right)^{i-1}\mathbf{g}_{\Delta c} \tag{22}$$

By developing the matrices and vectors in this expression, it can be shown that this is equivalent to

$$Cov_t\left(\Delta c_{t+i}, R_{t+1}\right) = a_{i,1}V_t\left(R_{t+1}\right) + a_{i,2}Cov_t\left(R_{t+1}, \Delta c_{t+1}\right) \tag{23}$$

where $a_{i,1}$ and $a_{i,2}$ are coefficients that are a linear combination of the coefficient in the matrix $\left(\mathbf{A}'\right)^{i-1}$. Plugging this result into equation (19) yields

$$E_t\left(R_{t+1}\right) = a_1 V_t\left(R_{t+1}\right) + a_2 Cov_t\left(\Delta c_{t+1}\right) \tag{24}$$

with $a_1 = \sum_{i=1}^{k} \alpha_i a_{i,1}$ and $a_2 = \sum_{i=1}^{k} \alpha_i a_{i,2}$.

3.2.2 External habit model

In external habit models, the representative agent does not take into account the influence of her consumption on her habit level (future or present). This kind of model has been initiated by Campbell and Cochrane (1999) and I will concentrate on their model. The utility function is the same as in the internal habit model presented in the previous section (cf. equation (15)). With external habits, we have a situation where the marginal utility of consumption is equal to

$$U'\left(C_t\right) = \left(C_t - H_t\right)^{-\gamma} \tag{25}$$

Defining the consumption surplus $S_t = \frac{C_t - H_t}{C_t}$ and using it to compute the logarithm of the SDF yields

$$m_{t+1} = -\gamma\Delta c_{t+1} - \gamma\Delta s_{t+1} \tag{26}$$

where $\Delta s_t = \ln S_t - \ln S_{t-1}$.

One particularity of Campbell and Cochrane's model is to define the evolution for the surplus s_t as follows [6]

$$s_{t+1} = \left(1-\omega\right)\bar{s} + \omega s_t + \lambda\left(s_t\right)\left(\Delta c_{t+1} - E_t\left(\Delta c_{t+1}\right)\right) \tag{27}$$

where $\bar{s}$ is the steady state surplus level, ω is a persistence parameter ($0 \leq \delta < 1$) and $\lambda\left(s_t\right)$ is the sensitivity of the surplus to unexpected changes in

[6] In the original model, Campbell and Cochrane use constant growth in consumption to compute $E_t\left(\Delta c_{t+1}\right)$. This generalisation is taken from Wachter (2005).

consumption, which is defined as

$$\lambda\left(s_t\right)=\begin{cases}\frac{1}{\bar{S}}\sqrt{1-2\left(s_t-\bar{s}\right)}-1 & \text{if } s_t<s_{\max}\\ 0 & \text{if } s_t\geq s_{\max}\end{cases} \tag{28}$$

where $\bar{S}=\exp\left(\bar{s}\right)$ and $s_{\max}=\bar{s}+\frac{1}{2}\left(1-\bar{S}^2\right)$.[7] With this dynamic, we have the following

$$\begin{aligned}m_{t+1}=&-\gamma\Delta c_{t+1}-\\&-\gamma\left(\left(1-\omega\right)\bar{s}+\left(\omega-1\right)s_t+\lambda\left(s_t\right)\left(\Delta c_{t+1}-E_t\left(\Delta c_{t+1}\right)\right)\right)\end{aligned} \tag{29}$$

Taking the conditional covariance of the log SDF with the expected excess return as equation (3) requires yields

$$E_t\left(R_{t+1}\right)=\gamma\left(1+\lambda\left(s_t\right)\right)Cov_t\left(R_{t+1},\Delta c_{t+1}\right) \tag{30}$$

With Campbell and Cochrane's model, the relative risk aversion is equal to $\gamma\left(1+\lambda\left(s_t\right)\right)$ and is thus time varying. Since sensitivity $\lambda\left(s_t\right)$ is negatively correlated with the surplus, then, the closer the agent is to its habit, the higher $\lambda\left(s_t\right)$ is and thus the higher her relative risk aversion.

The constrained model can be estimated by simultaneously estimating the VAR-GARCH-M together with equation (27). The resulting sensitivity $\lambda\left(s_t\right)$ is used for the varying coefficient associated to $Cov_t\left(R_{t+1},\Delta c_{t+1}\right)$ in the VAR-GARCH-M model.

3.3 Loss aversion functions

Utility functions with loss aversion are based on the Cumulative Prospect Theory developed by Kahneman and Tversky (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992). In this framework, utility is a function of the difference between consumption and a *reference level.*[8] The particularity of loss aversion functions is that losses have a bigger impact on utility than gains of the same magnitude. In this paper, I study two types of loss aversion utility functions: one in which the reference level is expected consumption

[7] Sensitivity $\lambda\left(s_t\right)$ is set equal to 0 when $s_t>s_{\max}=\bar{s}+\frac{1}{2}\left(1-\bar{S}^2\right)$ to ensure a positive term in the square root.

[8] The reference level is different from the habit level, which is defined as a subsistance level. Consumption may fall below the reference level in loss aversion models, whereas it has to stay above the subsistance level in habit formation models.

and one in which the reference level is based on past consumption. Note that in the original papers Benartzi and Thaler (1995) and Barberis, Huang, and Santos (2001) use a loss aversion function based on gains and losses in *financial wealth*. Barberis, Huang, and Santos (2001) justify the use of wealth instead of consumption as a way to capture "feelings unrelated to consumption" (p. 6). To remain close to the C-CAPM framework, in which an agent's utility is only influenced by her consumption, I use a loss aversion function defined on consumption rather than on financial wealth.

Following Köszegi and Rabin (2006), I use the general class of loss aversion function given by

$$U(C) = \sum_{i=0}^{\infty} \beta^i \left(\delta V(C_{t+i}) + (1-\delta) W(V(C_{t+i}) - V(X_{t+i}))\right) \tag{31}$$

with $\delta \in [0, 1]$. The utility of consumption in each period has two components: the first component $V(C_{t+i})$ is traditional "consumption utility". The second component $W(V(C_{t+i}) - V(X_{t+i}))$ is "gain-loss utility", which is the utility derived from the deviation of consumption C_{t+i} from a reference level X_{t+i}. With this function, in addition to the utility of her consumption, the agent can obtain "extra" utility if she consumes more than her reference level (gain in consumption utility). Similarly, if the agent consumes less than her reference level (loss in consumption utility), she will get less utility. Typically, with loss aversion, losses are more "painful" than gains and thus a loss generates a greater drop in utility than the increase in utility that would be derived from a gain of the same size.

In the next steps, I assume that the reference level is similar to the external habit in the sense that agents do not take into account the effect of their consumption on their present and future reference levels (i.e. $\partial X_{t+i}/\partial C_t = 0$). With this assumption, the marginal utility of consumption is equal to

$$U'(C_t) = V'(C_t)\left(\delta + (1-\delta) W'(V(C_t) - V(X_t))\right) \tag{32}$$

Using a first-order Taylor approximation of the logarithm of this expression, we obtain (see Appendix A)

$$u'(C_t) = k + v'(C_t) + qw'(V(C_t) - V(X_t)) \tag{33}$$

where k and q are linearisation parameters.

I assume that the consumption utility is the traditional power utility function

$(V(C_t) = C_t^{1-\sigma}/(1-\sigma))$ and that the gain-loss utility takes the form

$$W(Z) = \begin{cases} \frac{Z^{1-\nu}}{1-\nu} & \text{if } Z \geq 0 \\ -\eta\frac{(-Z)^{1-\nu}}{1-\nu} & \text{if } Z < 0 \end{cases} \tag{34}$$

where $\eta > 1$ and Z is the gain or loss of the agent. The parameter ν is the degree of loss aversion. The greater it is, the larger is the decrease in utility associated with a loss. Since $\eta > 1$, the utility gain associated with a gain Z is smaller than the utility loss associated with a loss of the same size Z. With this specification, we find that the logarithm of the *marginal* gain-loss utility is

$$w'(Z) = \begin{cases} -\nu \ln Z & \text{if } Z \geq 0 \\ \ln\eta - \nu\ln(-Z) & \text{if } Z < 0 \end{cases} \tag{35}$$

In our case, $Z_t = V(C_t) - V(X_t)$ and with a first order Taylor approximation of its logarithm, we find that (see Appendix C)

$$u'(C_t) = \begin{cases} k^+ + q^+ c_t + p^+ x_t \text{ if } c_t \geq x_t \\ k^- + q^- c_t + p^- x_t \text{ if } c_t < x_t \end{cases} \tag{36}$$

where ks, qs and ps are linear combination of the utility functions parameters and the linearisation parameters.

With this linear function, it is possible to compute the log SDF. The SDF takes two different forms, depending on the initial level of consumption. If the initial level of consumption is below the reference point $(c_t < x_t)$, then

$$m_{t+1} = \begin{cases} \Delta k + q^+\Delta c_{t+1} + p^+\Delta x_{t+1} + \Delta q c_t - \Delta p x_t & \text{if } c_{t+1} \geq x_{t+1} \\ q^-\Delta c_{t+1} + p^-\Delta x_{t+1} & \text{if } c_{t+1} < x_{t+1} \end{cases} \tag{37}$$

where $\Delta k = k^+ - k^-$, $\Delta q = q^+ - q^-$ and $\Delta p = p^+ - p^-$. If the initial level of consumption is above the reference point $(c_t > x_t)$, then

$$m_{t+1} = \begin{cases} q^+\Delta c_{t+1} + p^+\Delta x_{t+1} & \text{if } c_{t+1} \geq x_{t+1} \\ -\Delta k + q^-\Delta c_{t+1} + p^-\Delta x_{t+1} - \Delta q c_t - \Delta p x_t & \text{if } c_{t+1} < x_{t+1} \end{cases} \tag{38}$$

To compute the covariance between R_{t+1} and m_{t+1}, we use the following relationship (see Appendix D)

$$Cov(X, BY) = \left(\pi\frac{E(Y)}{V^{1/2}(Y)} + \Pi\right) Cov(X, Y) \tag{39}$$

where X and Y are two variables that are jointly normally distributed and B is a binary variable that take the value B^{+} and B^{-} in the event of gain and loss, respectively. Furthermore, $\pi = \phi\left(B^{+} - B^{-}\right)$ and $\Pi = \frac{\Phi^2+\phi^2}{\Phi}B^{-} + \frac{(1-\Phi)^2+\phi^2}{1-\Phi}B^{+}$ where Φ is the probability of observing a loss and ϕ is the probability of observing neither a gain nor a loss. With this relationship, it is possible to compute the covariance with the SDF in both situations (with initial consumption below or above the reference level). The covariance with the constants is equal to zero since that case is equivalent to $Y = 1$. Furthermore, as the reference level for the next period is known at time t (since it is equal to expected consumption or defined with respect to past consumption), its conditional covariance with R_{t+1} is also equal to 0. Therefore, in both cases, we have

$$Cov_t\left(R_{t+1}, m_{t+1}\right) = \left(\pi_t \frac{E_t\left(\Delta c_{t+1}\right)}{V_t^{1/2}\left(\Delta c_{t+1}\right)} + \Pi_t\right) Cov_t\left(R_{t+1}, \Delta c_{t+1}\right) \tag{40}$$

where $\pi_t = \phi_t \Delta q$ and $\Pi_t = \frac{\Phi_t^2+\phi_t^2}{\Phi_t}q^{-} + \frac{(1-\Phi_t)^2+\phi_t^2}{1-\Phi_t}q^{+}$ where Φ_t is the probability of observing a loss at time $t+1$ and ϕ_t is the probability of observing neither a gain or a loss.

Using this equation, we have

$$E_t\left(R_{t+1}\right) = -\psi_t Cov_t\left(R_{t+1}, \Delta c_{t+1}\right) \tag{41}$$

$$\text{with } \psi_t = \pi_t \frac{E_t\left(\Delta c_{t+1}\right)}{V_t^{1/2}\left(\Delta c_{t+1}\right)} + \Pi_t \tag{42}$$

The probability Φ_t that consumption falls below the reference level plays a significant role here since it determines the parameter ψ_t. To compute this probability, I use two different reference levels. In the first model (Loss aversion A), the reference level is equal to expected consumption

$$X_{t+1} = E_t\left(C_{t+1}\right) \tag{43}$$

The logarithm of expected consumption can be approximated by a second order Taylor approximation, which gives

$$x_{t+1} = c_t + E_t\left(\Delta c_{t+1}\right) + \frac{1}{2}V_t\left(\Delta c_{t+1}\right) \tag{44}$$

With this we can compute the probability Φ_t that the consumption level will fall below the reference level:

$$\Pr\left(c_{t+1} < x_{t+1}\right) = \Pr\left(\varepsilon_{t+1}^{\Delta c} < \frac{1}{2}V_t\left(\Delta c_{t+1}\right)\right) = \Phi\left(\frac{1}{2}V_t^{1/2}\left(\Delta c_{t+1}\right)\right) \tag{45}$$

where $\Phi()$ is the cumulative normal distribution.

In the second model (Loss aversion B), the other reference point is defined as a function of past consumption levels such that

$$X_{t+1} = C_t^{\varphi} X_t^{1-\varphi} \tag{46}$$

With this definition, we can derive the dynamic of the difference $d_t = c_t - x_t$ between the log consumption and the reference level

$$d_t = (1-\varphi)\, d_{t-1} + \Delta c_t \tag{47}$$

With this, the probability Φ_t that the consumption level will fall below the reference level is

$$\begin{aligned} \Pr(c_{t+1} < x_{t+1}) &= \Pr\left(\varepsilon_{t+1}^{\Delta c} < -(1-\varphi)\, d_t - E_t(\Delta c_{t+1})\right) \qquad (48) \\ &= \Phi\left(-\frac{(1-\varphi)\, d_t + g'_{\Delta c}\mathbf{A} y_t}{V_t^{1/2}(\Delta c_{t+1})}\right) \end{aligned}$$

4 Empirical results

4.1 Data

The data set contains monthly data on US consumption, prices and excess stock returns. The data are provided by Datastream. The excess return is the difference between the S&P500 index monthly return and the secondary market return on 3-month treasury bills. The real consumption growth rate Δc_t is computed using the US personal consumption expenditure index and the consumption price index. I examine the period between January 1965 and March 2007 (507 observations for each series).

4.2 Estimation method for the VAR-GARCH-M

As seen in section 2.2, the VAR-GARCH-M is an appropriate choice for modelling the joint dynamic implied by the no-arbitrage condition of the C-CAPM model. However, it has the disadvantage of being heavily parametrised. To limit the number of coefficients, I include only one lag for each variable of the model, which corresponds to a VAR(1)-GARCH(1,1)-M(1) model. Furthermore, I assume that the consumption growth rate Δc_t depends only on its past values and on past excess returns. In other words, its dynamic does not include a "in-mean" term, which is equivalent to setting the last two rows of matrix B in equation (4) equal to 0. In addition, the parameters of the variance-covariance dynamic must fulfill a number of conditions in order to ensure a positive definite covariance matrix. In order to achieve this, I apply the the BEKK specification (Engle and Kroner, 1995) to model the multivariate GARCH dynamic. The model can then be estimated by maximising the log likelihood function. [9]

I estimate three versions of the VAR-GARCH-M. The first one (Model 1) is the model in equation (4) and (5) with $y_t' = \begin{bmatrix} R_t \ \Delta c_t \end{bmatrix}$. This model is used to check whether the constraints of the constant relative risk aversion and the internal habit models are verified by the empirical data. The external habit model (Model 2) is a slightly modified version of Model 1. Firstly, as equation (30) shows, the covariance between the excess returns and the consumption growth enters into the excess return dynamic with a time varying coefficient. To take this into account, one term is added to equation (4)

$$y_t = \mu + Ay_{t-1} + Bh_{t-1} + De_{t-1} + fw_t + \varepsilon_t \tag{49}$$

where $w_t' = \lambda(s_{t-1}) \begin{bmatrix} h_{t-1}^{12} \ e_{t-1}^{12} \end{bmatrix}$ and f is a vector of coefficients. Since I have assumed that Δc_t is independent from the covariance term, I set the last row of f equal to zero. Sensitivity $\lambda(s_{t-1})$ is computed with equation (28). The surplus dynamic in equation (27) is estimated simultaneously with the VAR-GARCH-M equations (4) and (5) using likelihood maximisation. [10]

Similarly, the third version of the VAR-GARCH-M (Model 3) estimated the loss aversion models. Firstly, I estimate the equation (41) using the following model

$$y_t = \mu + Ay_{t-1} + Bh_{t-1} + De_{t-1} + \upsilon v_t + \varepsilon_t \tag{50}$$

where $v_t' = \left(\pi_t E_t(\Delta c_{t+1}) / V_t^{1/2}(\Delta c_{t+1}) + \Pi_t\right) \begin{bmatrix} h_{t-1}^{12} \ e_{t-1}^{12} \end{bmatrix}$ with $\pi_t = \phi_t(q_1 - q_2)$ and $\Pi_t = q_1\left(\Phi_t^2 + \phi_t^2\right)/\Phi_t + q_2\left((1-\Phi_t)^2 + \phi_t^2\right)/(1-\Phi_t)$. As in Model 2, the second row of the coefficient vector υ is set equal to zero. In the Loss aversion A model (expected consumption as reference level), probability Φ_t and ϕ_t are simply

$$\Phi_t = \Phi\left(\frac{1}{2}\left(h_t^2\right)^{1/2}\right) \text{ and } \phi_t = \phi\left(\frac{1}{2}\left(h_t^2\right)^{1/2}\right) \tag{51}$$

Similarly to the surplus dynamic equation in Model 2, iIn the Loss aversion B model (reference level as function of past consumption), the dynamic of

[9] See Hamilton (1994) for a more complete introduction to multivariate GARCH models and their estimation with maximum likelihood methods.

[10] Technically, the estimation of the surplus dynamic equation (30) is similar to the estimation of the variance dynamic in a GARCH model.

Table 1
Estimated unconstrained model 1

	R_t	Δc_t	V_t^1	V_t^2	Cov_t
c	-0.0064	0.0029**	7.56e-5**	1.50e-5*	0.98e-5*
R_{t-1}	-0.0275	0.0251**			
Δc_{t-1}	0.4062	-0.2063**			
V_{t-1}^1	3.8866		0.8681**		
V_{t-1}^2	258.5174*			0.2908**	
Cov_{t-1}	-114.9734**				0.5024**
$\left(\varepsilon_{t-1}^R\right)^2$	-1.1415		0.0918**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$	15.9545			0.2596**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	2.7790				0.1544**
Log likelihood	2823.608				

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$ and $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$

the difference between consumption and the reference level d_t is estimated simultaneously with equation (47). Probability Φ_t and ϕ_t are then

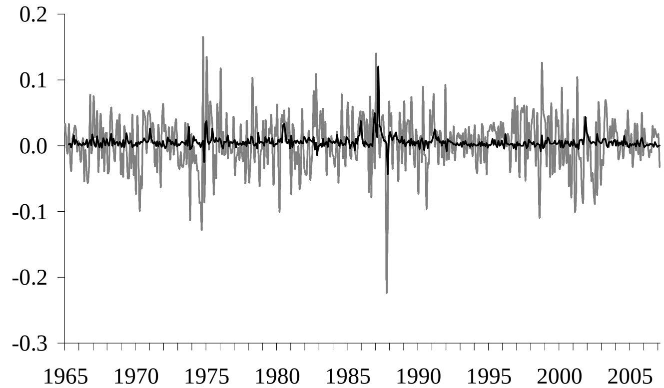
$$\Phi_t = \Phi\left(-\frac{(1-\eta)\,d_t + g'_{\Delta c}\mathbf{A}y_t}{(h_t^2)^{1/2}}\right) \text{ and } \phi_t = \phi\left(-\frac{(1-\eta)\,d_t + g'_{\Delta c}\mathbf{A}y_t}{(h_t^2)^{1/2}}\right) \quad (52)$$

4.3 Estimation of the joint dynamic

The estimation of the unconstrained Model 1 is displayed in Table 1. Figure 2 shows the observed excess returns and the expected excess return, given Model 1. Most of the coefficients of the model are not significant, with the exception of those associated to past consumption growth variance and to past covariance between consumption and excess returns. Note that the latter is significantly negative. The parameters of the consumption growth rate dynamic, conditional variances and conditional covariance are all significant. The excess return variance has the strongest inertia of the three second moments, followed by the covariance. Figure 3 shows the estimated second moments of excess returns and the consumption growth rate.

An important observation about the plausibility of the different models can already be made by looking at the joint dynamic coefficients. Note that the

Fig. 2. Excess Returns



Grey line: observed excess returns. Black line: estimated excess returns with Model 1.

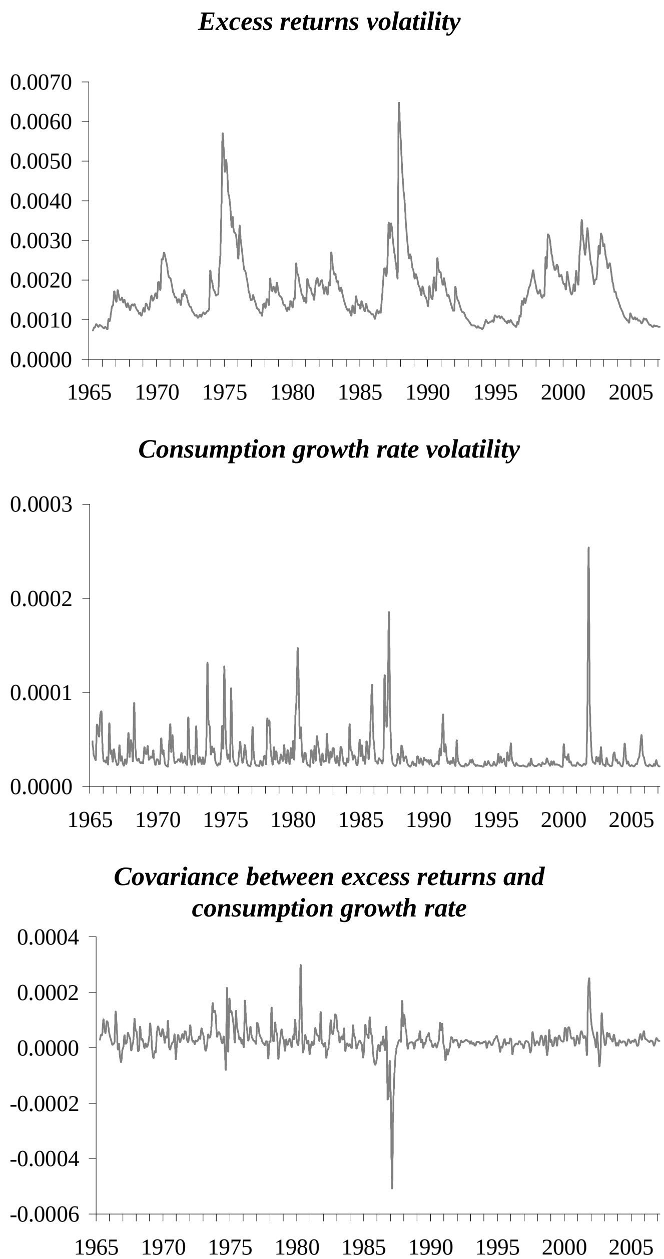
past conditional covariance between excess returns and consumption growth rates enters the equation with a significant negative coefficient and that the coefficient associated to the product of past residuals is not significant. Since the conditional covariance is a positive function of both terms, it suggests that the conditional covariance is inversely related to expected excess returns, as documented by Duffee (2005). The relationship is the opposite when unconditional moments are used. A negative correlation between conditional covariance and expected excess returns contradicts the results obtained from models based on power utility or external habit. Such correlation is, however, possible for some particular configurations of the external habit coefficients or for loss aversion utility. In the loss aversion case, this can be explained by the fact that loss averse agents can be risk lovers when they expect losses in consumption. Specifically, since losses cause negative utility, loss averse agents are ready to take more risk to avoid them. Risk-loving agents (or at least "occasional" risk loving agents) can then explain the observed negative correlation between the expected return and the conditional covariance.

4.4 Constant relative risk-aversion model

The estimated coefficients of the CRRA model are presented in Table 2. Not surprisingly, the coefficient of consumption growth and of the second moments are very similar to those of the unconstrained model.

With the VAR-GARCH-M approach, the estimated coefficient of relative aversion is approximately equal to 42. This value is in line with other previous studies and is well above the values that are traditionally considered as admissible. However, some authors claim that such a value might still be compatible with a representative agent with CRRA. Hens and Wohrmann (2006), for example,

Fig. 3. Estimated conditional second moments



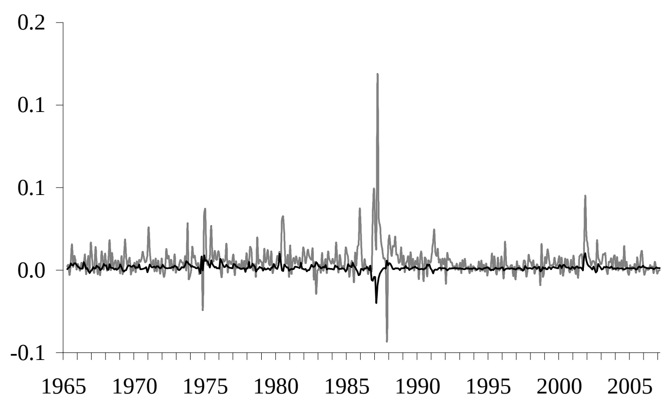
show that when mental accounting is applied to the CRRA model (as is normally done in experiments), an estimated CRRA higher than 10 is compatible with experimental results. The methodology proposed here offers another way to test the validity of the CRRA model, by comparing the theoretical and the observed evolution, rather than by assessing the plausibility of the coefficient value. The unconstrained and the constrained expected excess returns are presented in Figure 4. A first visual comparison suggests that both paths are quite different. In other words, the factors included in the unconstrained model but not in the CRRA model seem to have an impact on the evolution

Table 2
Estimated constant relative risk aversion model

	R_t	Δc_t	V_t^1	V_t^2	Cov_t
c		0.0029**	9.28e-5**	1.30e-5*	1.02e-5*
R_{t-1}		0.0229**			
Δc_{t-1}		-0.2050**			
σ	41.9390**				
V_{t-1}^1			0.8829**		
V_{t-1}^2				0.3620**	
Cov_{t-1}					0.5654**
$\left(\varepsilon_{t-1}^R\right)^2$			0.0650**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$				0.2492**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$					0.1273**
Log likelihood	2812.971				

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$ and $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$.

Fig. 4. Estimated excess returns with CRRA



Grey line: Unconstrained model 1. Black line: Constant relative risk aversion model.

of expected excess returns. Wether this impact is statistically significant is formally tested in section 4.9. A few statistics already give us some indication. The rank correlation[11] between the expected excess returns given by

[11] The rank correlation (Spearman correlation) is similar to the traditional correlation. It measure the link between two variables. The difference between traditional and rank correlations is that the former indicate the strength of the *linear* link

the unrestricted model and those given by the CRRA model is -0.05, which is negative but not significant. More worrying, when we restrict the sample to the months when the excess return is more extreme (inferior to the first quartile or superior to the third quartile), the rank correlation becomes significant and equal to -0.13. This means that the CRRA model tends to gives expected returns in the wrong direction when the excess returns (positive or negative) are large by historical standards.

4.5 Internal habit formation

The estimated coefficients of the internal habit formation model are presented in Table 3. The coefficients of the consumption growth rate dynamic and the second moments are similar to those of the unconstrained model. As opposed to the CRRA model, the coefficient associated with the covariance (α_2) is non significant (and negative). In this model, the excess return variance has a significant positive impact. The higher the excess return variance, the higher the expected return. This positive effect between excess return and its variance is well known from GARCH-in-mean analysis.

Figure 5 compares the expected excess returns of the unrestricted model with the excess return of the internal habit formation model. With this model, the rank correlation between them is significant and equal to 0.25. This implies that the excess returns generated with the internal habit formation model move in the same direction as the expected excess returns. Since the CRRA is a constrained version of the internal habit model (with $\alpha_1 = 0$), we can test whether the internal habit formation model significantly improves the CRRA model with a Wald test. The null hypothesis ($\alpha_1 = 0$) is rejected at a 1% confidence level. All these facts provide some indications that the internal habit formation model improves the traditional CRRA model.

4.6 External habit formation

As explained in Section 4.2, the unconstrained model for the external habit model is slightly different from Model 1, in order to incorporate a time-varying coefficient associated with covariance. The results of both the unconstrained and the constrained models for the external habit formation model are presented in Table 4. The coefficients of the consumption growth dynamic and the second moments dynamics are very similar to Model 1 in both models. Contrary to Model 1, in the excess return equation, the coefficient associated with

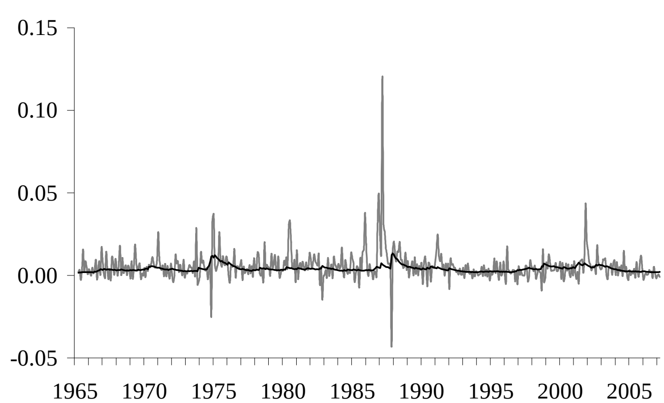
between two variables whereas the latter gives the strength of the link without assuming any particular form for this link.

Table 3
Estimated internal habit model

	R_t	Δc_t	V_t^1	V_t^2	Cov_t
c		0.0029**	6.96e-5**	1.34e-5*	1.03e-5*
R_{t-1}		0.0238**			
Δc_{t-1}		-0.2008**			
α_1	2.2691**				
α_2	-0.0887				
V_{t-1}^1			0.8890**		
V_{t-1}^2				0.3512**	
Cov_{t-1}					0.5587**
$\left(\varepsilon_{t-1}^R\right)^2$			0.0740**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$				0.2490**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$					0.1358**
Log likelihood	2816.191				

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$ and $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$

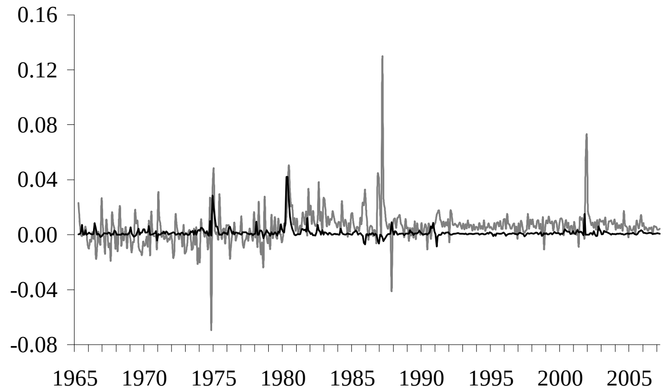
Fig. 5. Estimated excess returns with the internal habit formation



Grey line: Unconstrained model 1. Black line: Internal habit formation model.

the past covariance is no longer significant. The coefficient of consumption growth variance remains significant. All other coefficients are not significant. The constrained model indicates that the estimated parameter γ of the habit utility function is 0.61, which is consistent with its theoretical value. Figure 6 displays the expected excess returns of the unconstrained and the constrained

Fig. 6. Estimated excess returns with external habit formation model



Grey line: Unconstrained model 2. Black line: External habit formation model.

models. The rank correlation between the two series is statistically significant and equal to -0.15. This implies that the excess returns generated by the external habit formation models move in the opposite direction to the expected excess returns. This is a first negative sign for the external habit model.

In addition to the results presented in Table 4 and Figure 6, the estimation of the external habit also provides us with an estimation of the surplus dynamic given by equation (27). The estimated parameters that are relevant for the surplus dynamic are

$$\omega = 0.9924 \text{ and } \bar{S} = 59.36\%$$

I find an average surplus equal to 59% of total consumption. Figure 7 gives the estimated evolution for the surplus generated by these parameters as well as the evolution of the sensitivity and its estimated distribution. The estimation shows that the surplus was the lowest between 1981 and 1983 with a minimum of 55.8% of total consumption in June 1982. The first half of the nineties is also a period in which the surplus was relatively low. Since then it has been situated below its mean but close to it. These results suggest that investors were the most risk-averse in 1981-1983 and in the first half of the nineties. Accordingly, during these period they required higher returns to compensate for risk. Note that, I observe no significant correlation between the coefficient of risk aversion and the covariance between consumption growth and excess returns. Thus, according to this estimation, risk aversion is independent of the risk associated with the stock market.

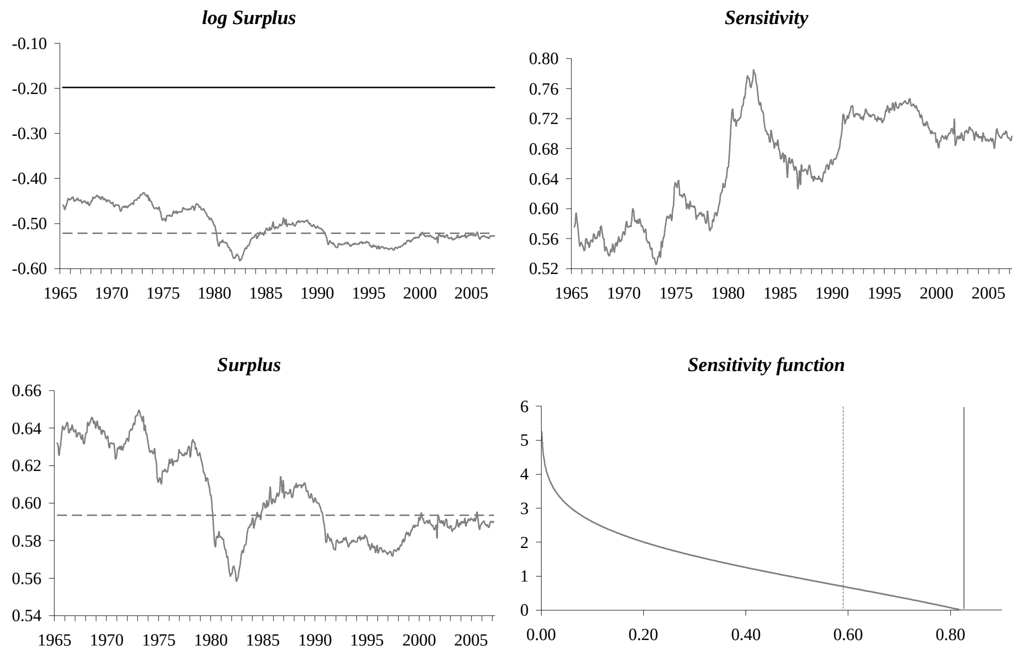
The estimated persistence factor ω estimated here is slightly higher than in Campbell and Cochrane (1999) (0.9924 vs. 0.9856). The estimated average surplus is about 10 times greater than than estimated by Campbell and Cochrane: 59.36% vs. 5.7%! One consequence of this difference is that the surplus estimated here moves much further away from the habit than in Campbell and

Table 4
Estimated external habit model

	R_t	Δc_t	V_t^1	V_t^2	Cov_t
Unconstrained model 2					
c	-0.0177	0.0029**	5.71e-5**	1.85e-5*	1.40e-5*
R_{t-1}	-0.0525	0.0261**			
Δc_{t-1}	0.5299	-0.2161**			
V_{t-1}^1	2.3552		0.8615**		
V_{t-1}^2	348.8674*			0.1815**	
Cov_{t-1t}	-1290.8180				0.3955**
$\left(\varepsilon_{t-1}^R\right)^2$	-1.3288		0.1110**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$	16.2469			0.2673**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	82.2660				0.1722**
λ	0.0164				
λCov_{t-1}	1830.7340				
$\lambda \varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	-115.7720				
Log likelihood	2832.693				
Constrained model					
c		0.0029**	8.42e-5**	1.24e-5**	0.77e-5**
R_{t-1}		0.0241**			
Δc_{t-1}		-0.2049**			
V_{t-1}^1			0.8782**		
V_{t-1}^2				0.3812**	
Cov_{t-1}					0.5785**
$\left(\varepsilon_{t-1}^R\right)^2$			0.0772**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$				0.2483**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$					0.1385**
γ	0.6142				
Log likelihood	2815.027				

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$, $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$ and $\lambda = \lambda(s_{t-1})$.

Fig. 7. Estimated suprlus and sensitivity



In each panel, the solid black line corresponds to the theoretical maximum surplus and the dashed grey line to the average surplus. The upper right panel shows the sensitivity corresponding to the estimated surplus. The lower right panel displays the function linking the sensitivity (vertical axis) to the surplus (horizontal axis).

Cochrane. With such a large gap, sensitivity is relatively stable and low. This implies that the price of risk generated by external habit formation model is also stable and low. This is in line with Duffee (2005), who argues that the price of risk in the external habit formation model does not vary enough to justify the large swings in the equity premium and dismisses Campbell and Cochrane's external habit formation model.

4.7 Loss-aversion model A (reference point based on expected consumption)

The estimation of the loss-aversion model with a reference point based on expected consumption is presented in Table 5. As for previous models, the coefficients for consumption growth and conditional second moments are similar to Model 1. In the unconstrained model, all coefficients are non-significant.

Figure 8 presents the expected excess returns given by the unconstrained model and by the restricted model. We observe a rank correlation of 0.06, which is not statistically significant. In addition, the estimation gives us the following parameters

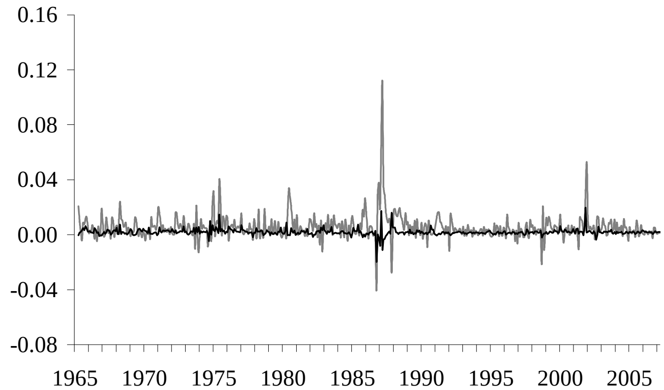
$$q^{-} = 112.29 \text{ and } q^{+} = -102.88$$

Table 5
Estimated loss aversion model A

	R_t	Δc_t	V_t^1	V_t^2	Cov_t
Unconstrained model 3					
c	-0.0048	0.0028**	6.90e-5**	1.76e-5	1.24e-5
R_{t-1}	-0.0136	0.0269**			
Δc_{t-1}	0.3605	-0.1849**			
V_{t-1}^1	3.8066		0.8620**		
V_{t-1}^2	272.9046			0.2149**	
Cov_{t-1}	-128.9434				0.4304**
$\left(\varepsilon_{t-1}^R\right)^2$	-1.3869		0.1032**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$	9.6495			0.2552**	.
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	9.6022				0.1623**
η_t	0.0002				
$\eta_t Cov_{t-1}$	2.7163				
$\eta_t \varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	-0.6421				
Log likelihood	2825.084				
Constrained model					
c		0.0028**	8.97e-5**	1.39e-5**	1.09e-5**
R_{t-1}		0.0221**			
Δc_{t-1}		-0.1933**			
V_{t-1}^1			0.8835**		
V_{t-1}^2				0.3445**	
Cov_{t-1}					0.5517**
$\left(\varepsilon_{t-1}^R\right)^2$			0.0636**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$				0.2312**	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$					0.1213**
Log likelihood	2813.880				

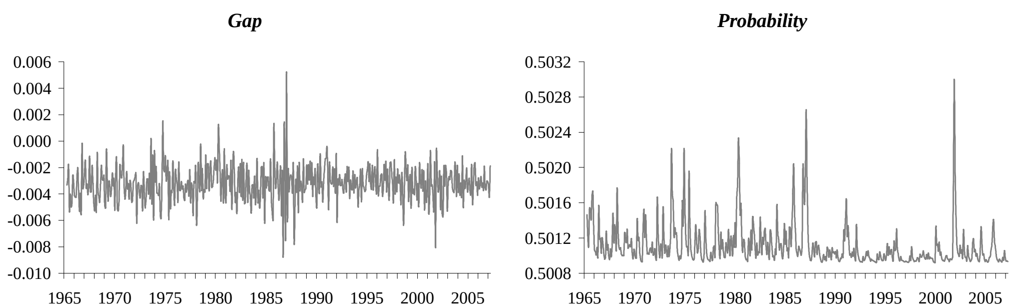
* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$ and $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$.

Fig. 8. Estimated excess returns with loss aversion A



Grey line: Unconstrained model 3. Black line: Loss aversion A.

Fig. 9. Gap with reference level and probability to fall under it (loss aversion A)



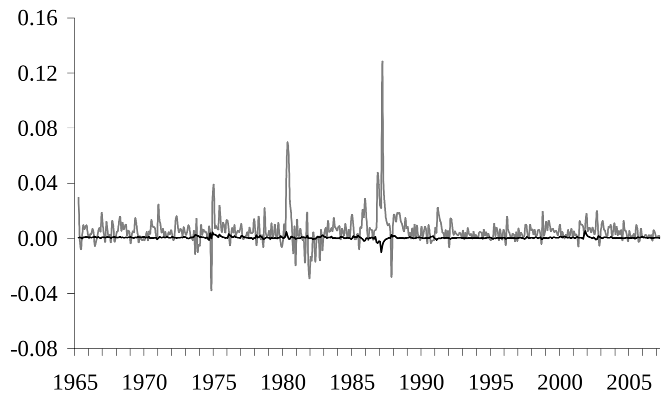
Left panel: gap between the consumption and its reference level. Right panel: probability for the consumption to be under its reference level in the next period.

Note however that these two coefficients are not statistically significantly different from each other, which makes it difficult to infer some value for the utility parameter ν (cf. Appendix C). With these coefficients it is possible to estimate the changes in the difference between the reference level and actual consumption and the probability to fall under the reference level in the next period (cf. Figure 9). We can see that this probability varies from one period to another, but stays very close to its mean of 50.11%.

4.8 Loss aversion model B (reference level based on past consumption)

The estimation of the loss-aversion model with a reference level based on past consumption is presented in Table 6. As for previous models, the coefficients of the consumption growth and conditional second moments are similar to Model 1. In the unconstrained model, all coefficients are non significant. The results of the constrained model are more disturbing because the coefficients

Fig. 10. Estimated excess returns with loss aversion B



Grey line: Unconstrained model 3. Black line: Loss aversion B.

are non significant, which is very different from the previous models. In all other models, we observe stability of the coefficient for the consumption growth and second moments dynamics. Here, the coefficients change significantly.

Figure 10 presents the expected excess returns given by the unconstrained model and by the restricted model. We observe a rank correlation of -0.07, which is statistically not significant. In addition, the estimation of the constrained model gives us the following parameters

$$q^{-} = 91.07 \text{ and } q^{+} = -38.40$$

These two coefficients are not significantly different in statistical terms from each other, which makes it difficult to infer a value for the utility parameter ν (cf. Appendix C). The unconstrained model give us $\varphi = 0.9593$, whereas the constrained model gives $\varphi \simeq 1$ which corresponds to a model where the reference level is equal to the present consumption.

4.9 Tests on the excess returns dynamic

As section 2.3 explains, it is possible to use a formal test to check if the evolution of the expected returns follows that predicted by one particular model. For this purpose, I test the set of restrictions, $\mathbf{g}'\mathbf{A} = 0$, using a Wald test and a Likelihood ratio test. The results are displayed in Table 7.

The results clearly show that the null hypothesis is strongly rejected for the CRRA model and for the external habit formation model. Therefore, the path implied by these models does not formally correspond to the observed evolution. The internal habit formation model and the loss aversion model with a reference level based on past consumption are accepted at a 1% confidence

Table 6
Estimated loss aversion model B

Unconstrained model 3					
	R_t	Δc_t	V_t^1	V_t^2	Cov_t
c	0.0149	0.0029**	6.70e-5**	1.84e-5	1.40e-5
R_{t-1}	-0.0223	0.0240**			
Δc_{t-1}	0.5159	-0.1943**			
V_{t-1}^1	3.4726		0.8566**		
V_{t-1}^2	223.8821			0.1874**	
Cov_{t-1}	-323.9910				0.4006**
$\left(\varepsilon_{t-1}^R\right)^2$	-0.8172		0.1111**		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$	0.3169			0.2609**	.
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	-43.0290				0.1702**
η_t	0.0011				
$\eta_t Cov_{t-1}$	-11.6878				
$\eta_t \varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$	-2.6755				
Log likelihood	2830.753				
Constrained model					
c		0.0035*	8.34e-5	3.68e-5	5.25e-5
R_{t-1}		-0.0243			
Δc_{t-1}		-0.1256			
V_{t-1}^1			0.8592		
V_{t-1}^2				0.2013	
Cov_{t-1}					0.4159
$\left(\varepsilon_{t-1}^R\right)^2$			0.1055		
$\left(\varepsilon_{t-1}^{\Delta c}\right)^2$				0.2568	
$\varepsilon_{t-1}^R \varepsilon_{t-1}^{\Delta c}$					0.1646
Log likelihood	2818.584				

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level. $V_t^1 = V_t(R_{t+1})$, $V_t^2 = V_t(\Delta c_{t+1})$ and $Cov_t = Cov_t(R_{t+1}, \Delta c_{t+1})$

Table 7
Test for the different models

	Wald test		Likelihood Ratio test	
	Test stat.	p-value	Test stat.	p-value
Constant relative risk aversion	24.1077	0.0022	21.2745	0.0065
Internal habit formation	17.1301	0.0166	14.8338	0.0382
External habit formation	29.1585	0.0021	35.3325	0.0002
Loss aversion A	17.8322	0.1209	22.4083	0.0332
Loss aversion B	23.3300	0.0251	24.4598	0.0176

* (**) denotes that the coefficient is significant at the 5% (1%) confidence level.

level by both tests. The loss aversion model with a reference level defined by the expected consumption is *not* rejected at a 10% confidence level by the Wald test and at a 1% confidence level by the Likelihood ratio test. On the basis of these tests, the loss aversion model with expected consumption seems to be the one which is the most adequate to the observed evolution of excess return. It is, however, still rejected by the likelihood test at a 5% confidence level. The internal habit formation model and the loss aversion model based on past consumption are the next plausible candidates, but they are both rejected at a 5% confidence level. Finally, the constant relative risk-aversion model and the external habit formation model are strongly rejected.

5 Conclusion

The results of the estimated linear SDF models show that the constant relative risk-aversion model and the external habit model are strongly rejected by the data. The theoretical excess returns generated by these models do not correspond to the observed evolution. I also find evidence that the conditional covariance between excess returns and consumption growth is negatively related to excess returns, which contradicts theses two models. This finding is in-line with Duffee (2005) results. Finally, the constraints implied by constant relative risk aversion and external habit formation are formally rejected by both the Wald and the likelihood ratio tests. The validity of the loss aversion model using past consumption as reference level is also doubtful. Although its restrictions cannot be rejected at a 1% confidence level, the values of the estimated parameters cast some doubt on it.

On the basis of these conclusion, two candidates remain for explaining the excess return path: the internal habit formation model and the loss-aversion model with expected consumption as reference level. Neither model can be rejected at a 1% confidence level and they have plausible parameters. My preference lies with the loss-aversion model because the Wald test does not reject its constraints at a 10% level, whereas the internal habit formation model is rejected at a 5% level by both tests. This suggests that loss-aversion models do play a role in the expected excess returns movements and that they might be a suitable answer to the equity premium puzzle. Consequently, the loss-aversion component should be included in the consumption-based model in this context. Note that the results presented here also confirm those of Chen and Ludvigson (2004), who find that the data are better described by internal habit formation than by external habit formation.

One interesting follow-up to this paper would be to use the VAR-GARCH-M framework to test the original specification of the loss-aversion model, where the loss function is defined on wealth rather than on consumption. If the change in wealth better explains the data, then we should remember Barberis, Huang, and Santos (2001) justification for using wealth, and study "feelings unrelated to consumption" (Barberis, Huang, and Santos, 2001, p. 6) to understand the factors driving the expected excess returns. Without looking too far away from the original consumption-based model, the results presented here suggest that consumption growth variance, i.e. the uncertainty about consumption, might play a role in the excess return evolution. This factor does not appear in any of the proposed models and thus might constitute a starting point for developing a more adequate model.

A Approximation for linear combinations of logarithms

A.1 Logarithm of a sum

Campbell, Lo, and MacKinlay (1997) show that we can approximate the logarithm of a sum by the sum of logarithms. First consider

$$\ln(A+B) = \ln A\left(1+\frac{B}{A}\right) = a + \ln\left(1+e^{b-a}\right) \tag{A.1}$$

where $a = \ln A$ and $b = \ln B$. The second part of this equation can be approximated by a standard Taylor approximation around its mean. Define $x = b - a$ and $f(x) = \ln(1+e^{x})$ The Taylor approximation yields $f(x) \approx f(\bar{x}) + f'(\bar{x})(x-\bar{x})$ with $f'(\bar{x}) = e^{\bar{x}}/(1+e^{\bar{x}})$. Plugging this into equation (A.1) gives the final result

$$\ln(A+B) \simeq \kappa + \rho a + (1-\rho)b \tag{A.2}$$

with $\rho = 1/\left(1+e^{\bar{x}}\right)$ and $\kappa = -\ln\rho - (1-\rho)\ln(1/\rho - 1)$. We have that $0 < \rho < 1$

A.2 Logarithm of a subtraction

Similarly for a subtraction, we have that

$$\ln(A-B) = \ln A\left(1-\frac{B}{A}\right) = a + \ln\left(1-e^{b-a}\right) \tag{A.3}$$

Define $g(x) = \ln(1-e^{x})$ which implies $g'(\bar{x}) = -\frac{e^{\bar{x}}}{1-e^{\bar{x}}}$ and thus

$$\ln(A-B) \simeq \varsigma + \psi a - (\psi-1)\, b \tag{A.4}$$

with $\psi = 1/\left(1-e^{\bar{x}}\right)$ and $\varsigma = -\ln\psi - (1-\psi)\ln(1-1/\psi)$. Note that this approximation is possible only if $A-B>0$, which implies that $b-a<0$ and thus $\psi > 1$.

B Linearisation of the internal habit marginal utility

From the definition of the habit level in equation (5), we can compute the derivative of the habit with respect to consumption:

$$\frac{\partial H_{t+i}}{\partial C_t} = \begin{cases} \theta_i H_{t+i} C_t^{-1} & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases} \tag{B.1}$$

Plugging this into equation (16)

$$U'(C_t) = (C_t - H_t)^{-\gamma} - \sum_{i=0}^{k} \delta^i \left(C_{t+i} - H_{t+i}\right)^{-\gamma} \theta_i H_{t+i} C_t^{-1} \tag{B.2}$$

The first order condition of the C-CAPM states that the marginal utility of consumption is equal to the Lagrangian of the maximisation problem. The Lagrangian is equal to the marginal utility of relaxing the budget constraints by one unit, which corresponds to the marginal utility of one additional unit of initial wealth. By definition, this marginal utility is always positive, which implies that the marginal utility of consumption is also positive and its logarithm exists. Taking the log of this difference yields (see Appendix A)

$$\ln U'(C_t) = \varsigma_1 - \gamma\psi_1 \ln(C_t - H_t) + (\psi_1 - 1)c_t - \tag{B.3}$$

$$-(\psi_1 - 1)\ln\sum_{i=0}^{k}\delta^i (C_{t+i} - H_{t+i})^{-\gamma}\theta_i H_{t+i}$$

The log of the sum in the last term can also be linearised with a Taylor first order approximation (see Appendix A). We have that

$$\ln\sum_{i=0}^{\infty}\delta^i (C_{t+i} - H_{t+i})^{-\gamma}\theta_i H_{t+i} = \tag{B.4}$$

$$= \kappa_1 + (1-\rho_1)(\ln\theta_0 - \gamma\ln(C_t - H_t) + h_t)$$
$$+\rho_1\ln\sum_{i=1}^{\infty}\delta^i (C_{t+i} - H_{t+i})^{-\gamma}\theta_i H_{t+i}$$

Solving this equation forward yields

$$\ln\sum_{i=0}^{\infty}\delta^i (C_{t+i} - H_{t+i})^{-\gamma}\theta_i H_{t+i} = \tag{B.5}$$

$$= \sum_{i=0}^{k}\left(\prod_{j=0}^{i-1}\rho_j\right)(\kappa_i + (1-\rho_i)(i\ln\delta + \ln\theta_i - \gamma\ln(C_{t+i} - H_{t+i}) + h_{t+i}))$$

Collecting all constant terms in κ'' yields

$$\ln\sum_{i=0}^{\infty}\delta^i (C_{t+i} - H_{t+i})^{-\gamma}\theta_i H_{t+i} = \kappa'' + \tag{B.6}$$

$$+\sum_{i=0}^{k}\left(\prod_{j=0}^{i-1}\rho_j\right)(1-\rho_i)(h_{t+i} - \gamma\ln(C_{t+i} - H_{t+i}))$$

Since, by definition, consumption is always greater than the habit, we can also approximate the logarithm of the difference between them

$$\ln(C_t - H_t) = \varsigma_2 + \psi_2 c_t - (\psi_2 - 1)h_t \tag{B.7}$$

Taking the logarithm of equation (17) gives

$$h_t = \theta_0 c_t + \theta_1 c_{t-1} + ... + \theta_k c_{t-k} = L(\boldsymbol{\theta}, \mathbf{c}_t) \tag{B.8}$$

where $L(\boldsymbol{\theta}, \mathbf{c}_t)$ is a linear combination of the vector $\boldsymbol{\theta}$ and $\mathbf{c}_t$ which collects the parameters $\theta_0, ..., \theta_k$ and the log consumption $c_t, ..., c_{t-k}$. Combing the two last equations gives

$$\ln(C_t - H_t) = \varsigma_2 + \psi_2 c_t - (\psi_2 - 1) L(\boldsymbol{\theta}, \mathbf{c}_t) \tag{B.9}$$

Plugging this results into equation (B.4) gives

$$\ln \sum_{i=0}^{\infty} \delta^i (C_{t+i} - H_{t+i})^{-\gamma} \theta_i H_{t+i} = \tag{B.10}$$

$$= \kappa' - \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} \rho_j \right) (1 - \rho_i) (\gamma \psi_2 c_{t+i} - (1 + \gamma(\psi_2 - 1)) L(\boldsymbol{\theta}, \mathbf{c}_{t+i}))$$

This equation tells us that $\ln \sum_{i=0}^{\infty} \delta^i (C_{t+i} - H_{t+i})^{-\gamma} \theta_i H_{t+i}$ is a linear function of c_{t+i} with for all $i \in [-k, k]$. This can be simply rewritten as

$$\ln \sum_{i=0}^{k} \delta^i (C_{t+i} - H_{t+i})^{-\gamma} \theta_i H_{t+i} = \kappa' + \sum_{i=-k}^{k} b_i c_{t+i} \tag{B.11}$$

Finally combining this equation with equation (B.3) also gives a linear function:

$$\ln U'(C_t) = \kappa + \sum_{i=-k}^{k} a_i c_{t+i} \tag{B.12}$$

which implies that

$$m_{t+1} = \sum_{i=-k+1}^{k+1} a_i \Delta c_{t+i} \tag{B.13}$$

C Linearisation of the loss aversion marginal utility

Rewriting equation (35) gives

$$w'(Z) = \begin{cases} -\nu \ln(V(C_t) - V(X_t)) & \text{if } C_t \geq X_t \\ \ln \eta - \nu \ln(V(X_t) - V(C_t)) & \text{if } C_t < X_t \end{cases} \tag{C.1}$$

Since both $V(C_t) - V(X_t)$ in the upper part of this equation and $V(X_t) - V(C_t)$ in the lower part of it are by definition positive, we can approximate them with a Taylor approximation (cf. Appendix A). We get that

$$\ln (V(C_t) - V(X_t)) = \varsigma_3 + \psi_3 \ln V(C_t) - (\psi_3 - 1) \ln V(X_t) \tag{C.2}$$

if $C_t \geq X_t$ and

$$\ln (V(X_t) - V(C_t)) = \varsigma_4 + \psi_4 \ln V(X_t) - (\psi_4 - 1) \ln V(C_t) \tag{C.3}$$

if $C_t < X_t$ with

$$\psi_3 = 1/\left(1 - \exp\left(\overline{\ln V(C_t) - \ln V(X_t)}\right)\right)$$

$$\psi_4 = 1/\left(1 - \exp\left(\overline{\ln V(X_t) - \ln V(C_t)}\right)\right)$$

where $\overline{\ln V(A) - \ln V(B)}$ is the average over all $A \geq B$.

Using the fact that $V()$ is a power utility function, we get that

$$\ln V(C_t) = (1-\sigma) c_t - \ln (1-\sigma) \tag{C.4}$$

$$\ln V(X_t) = (1-\sigma) x_t - \ln (1-\sigma) \tag{C.5}$$

and

$$\ln (V(C_t) - V(X_t)) = \varsigma_3' + (1-\sigma)(\psi_3 c_t - (\psi_3 - 1) x_t) \tag{C.6}$$
$$\ln (V(X_t) - V(C_t)) = \varsigma_4' + (1-\sigma)(\psi_4 x_t - (\psi_4 - 1) c_t) \tag{C.7}$$

Plugging these last equations into equation (C.1) gives

$$u'(C_t) = \begin{cases} k^+ + q^+ c_t + p^+ x_t \text{ if } c_t \geq x_t \\ k^- + q^- c_t + p^- x_t \text{ if } c_t < x_t \end{cases} \tag{C.8}$$

where

$$\begin{aligned} k^+ &= k - \theta q \varsigma_3' \\ q^+ &= -(\sigma + \theta q (1-\sigma) \psi_3) \\ p^+ &= \theta (1-\sigma) q (\psi_3 - 1) \\ k^- &= k - \theta q \varsigma_4' + q \ln \lambda \\ q^- &= -(\sigma + \theta q (1-\sigma)(\psi_4 - 1)) \\ p^- &= -\theta q (1-\sigma) \psi_4 \end{aligned}$$

D Covariance with loss aversion function

Let us define X and Y as two joint normally distributed random variables and Z as a binary variable such as $Z = 1$ if $Y < \bar{Y}$ and 0 otherwise. The variable B is a binary variable that takes value B^- if $Z = 1$ and B^+ otherwise. Let us define Φ as the probability that $Z = 1$ and ϕ as the probability that $Y = \bar{Y}$.

Let us start with the identity

$$Cov\left(X, BY\right) = E\left(Cov\left(X, BY|\, Z\right)\right) + Cov\left(E\left(X|\, Z\right), E\left(BY|\, Z\right)\right) \tag{D.1}$$

The first part of the right hand side of equation (D.1) is equal to

$$E\left(Cov\left(X, BY|\, Z\right)\right) = \left(\Phi B^- + (1-\Phi)\, B^+\right) Cov\left(X, Y\right) \tag{D.2}$$

To compute the second part, first note that

$$E\left(BY|\, Z\right) = E\left(B|\, Z\right) E\left(Y|\, Z\right) + Cov\left(B, Y|\, Z\right) \tag{D.3}$$

Since B is a constant when Z is known, then $Cov\left(B, Y|\, Z\right) = 0$ and

$$Cov\left(E\left(X|\, Z\right), E\left(BY|\, Z\right)\right) = Cov\left(E\left(X|\, Z\right), E\left(B|\, Z\right) E\left(Y|\, Z\right)\right) \tag{D.4}$$

Then define the two residuals $e^x = X - E\left(X\right)$ and $e^y = Y - E\left(Y\right)$, which implies that

$$E\left(X|\, Z\right) = E\left(X\right) + E\left(e^x|\, Z\right) \tag{D.5}$$

$$E\left(Y|\, Z\right) = E\left(Y\right) + E\left(e^y|\, Z\right) \tag{D.6}$$

Plugging these definitions into the previous equations yields

$$Cov\left(E\left(X|\, Z\right), E\left(BY|\, Z\right)\right) = \tag{D.7}$$

$$= Cov\left(E\left(X\right) + E\left(e^x|\, Z\right), E\left(B|\, Z\right)\left(E\left(Y\right) + E\left(e^y|\, Z\right)\right)\right)$$

Since $E\left(X\right)$ and $E\left(Y\right)$ are constant, this equation becomes

$$Cov\left(E\left(X|\, Z\right), E\left(BY|\, Z\right)\right) = \tag{D.8}$$

$$= E\left(Y\right) Cov\left(E\left(e^x|\, Z\right), E\left(B|\, Z\right)\right) + Cov\left(E\left(e^x|\, Z\right), E\left(B|\, Z\right) E\left(e^y|\, Z\right)\right)$$

Furthermore, using the fact that $Cov\left(a, b\right) = E\left(ab\right) - E\left(a\right) E\left(b\right)$ and that, by the law of iterated expectations, $E\left(E\left(e^x|\, Z\right)\right) = E\left(e^x\right) = 0$, we can simplify the previous equation to

$$Cov\left(E\left(X|\,Z\right),E\left(BY|\,Z\right)\right)= \tag{D.9}$$

$$=E\left(Y\right)E\left(E\left(e^{x}|\,Z\right)E\left(B|\,Z\right)\right)+E\left(E\left(e^{x}|\,Z\right)E\left(B|\,Z\right)E\left(e^{y}|\,Z\right)\right)$$

Developing the expectation term, we get

$$Cov\left(E\left(X|\,Z\right),E\left(BY|\,Z\right)\right)=\Phi B^{-}E\left(e^{x}|\,1\right)\left(E\left(Y\right)+E\left(e^{y}|\,1\right)\right)+ \tag{D.10}$$

$$+\left(1-\Phi\right)B^{+}E\left(e^{x}|\,0\right)\left(E\left(Y\right)+E\left(e^{y}|\,0\right)\right)$$

For the next step, let us assume that X and Y jointly normally distributed. This implies that

$$E\left(e^{x}|\,e^{y}\right)=\frac{Cov\left(e^{x},e^{y}\right)}{V\left(e^{y}\right)}e^{y} \tag{D.11}$$

Taking the conditional expectation on Z yields

$$E\left(e^{x}|\,Z\right)=\frac{Cov\left(e^{x},e^{y}\right)}{V\left(e^{y}\right)}E\left(e^{y}|\,Z\right) \tag{D.12}$$

The last term of this equation is the expectation of a truncated normal distribution and is equal to (see Greene, 2000, p. 899)

$$E\left(e^{y}|\,Z\right)=\begin{cases}-V^{1/2}\left(e^{y}\right)\frac{\phi}{\Phi}\text{ if }Z=1\\ V^{1/2}\left(e^{y}\right)\frac{\phi}{1-\Phi}\text{ if }Z=0\end{cases} \tag{D.13}$$

which implies by plugging equations (D.12) and (D.13) into equation (D.10)

$$Cov\left(E\left(X|\,Z\right),E\left(BY|\,Z\right)\right)= \tag{D.14}$$

$$=\phi\left(\frac{B^{+}-B^{-}}{V^{1/2}\left(e^{y}\right)}E\left(Y\right)+\phi\left(\frac{B^{+}}{1-\Phi}+\frac{B^{-}}{\Phi}\right)\right)Cov\left(e^{x},e^{y}\right)$$

Finally, combining equations (D.2) and (D.14), we get the final expression for the covariance in equation (D.1)

$$Cov\left(X,BY\right)= \tag{D.15}$$

$$=\left(\phi\frac{B^{+}-B^{-}}{V^{1/2}\left(e^{y}\right)}E\left(Y\right)+\frac{\Phi^{2}+\phi^{2}}{\Phi}B^{-}+\frac{\left(1-\Phi\right)^{2}+\phi^{2}}{1-\Phi}B^{+}\right)Cov\left(e^{x},e^{y}\right)$$

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Part III

Common Exposure and Systemic Risk in the Banking Sector ⋆

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Abstract

In the last decades, international financial integration has markedly increased. This paper analyses the impact of these developments on banks' common exposure to shocks and on banking sector systemic risk. Theoretically, this impact is ambiguous. Empirically, we find that banks' common exposure to shocks has significantly decreased until 2000 and rapidly increased afterwards. Systemic risk follows an entirely different pattern with two peaks in 1998 and in 2002-2003. We provide evidence that systemic risk is mainly driven by banks' individual risk-taking and that, contrary to widespread belief, higher common exposures to risk do not necessarily induce higher systemic risk.

Key words: Systemic risk, Co-movements, Banking sector.

JEL classification: F36, G15, G21.

⋆ The opinions expressed herein are those of the authors and do not necessarily reflect the views of the Swiss National Bank.

1 Introduction

Worldwide, the banking sector has gone through profound transformations in the last few decades. Technical progress in financial engineering and communications technologies as well as global deregulation policies have significantly modified the international financial landscape. In Europe, the launch of the euro has also helped to accelerate these changes (Baele, Ferrando, Hördahl, Krylova, and Monnet, 2004). An obvious outcome of these developments is a quickening in the pace with which international financial markets are being integrated. Financial institutions – large banks in particular – now benefit from much easier access to a wider range of markets and financial instruments. These developments have led some observers to worry about a possible increase in the systemic risk of the banking sector. Their main fear is that financial integration increases direct inter-linkages between banks and also causes banks to compete more and more on the same markets, exposing them to the same risk factors. This common exposure to risks means that a downward shock could impact on most of the banks simultaneously and thus trigger a systemic crisis in the banking sector.

Although this concern is well founded, from a theoretical point of view it is not clear that financial integration automatically induces higher common exposure to shocks. Neither is it clear that higher common exposure causes an increase in systemic risk. On the one hand, a wider range of markets and financial tools offers banks an opportunity to differentiate themselves by implementing their own specific business strategy.[1] Adopting different strategies reduces the common exposure of two banks to shocks. Access to new markets and new financial techniques also offers financial institutions more opportunities to manage and diversify their risks, which is beneficial in terms of systemic risk. On the other hand, easy access and low entry costs are likely to increase the number of banks competing on the same market. Furthermore, intensified cross-border linkages between banks increase the risk of contagion. Both effects bring about an increase in banks' common exposure to shocks and are relatively negative as regards systemic stability. To sum up, a higher level of financial integration gives rise to both positive and negative effects in terms of common exposure to shocks and systemic risk. The net effect is ambiguous. Furthermore, the link between common exposure and systemic risk is also ambiguous. Indeed, a higher common exposure to shocks does not necessarily imply higher systemic risk. An increase in common exposure can, for example,

[1] Hellwig (1998) shows that banks have an incentive to specialise rather than diversify in order to reduce their monitoring costs. Winton (1999) and Stomper (2006) show that both specialised and diversified banks can coexist in equilibrium. Fecht, Grüner, and Hartmann (2007) show that the improvements in risk sharing induced by integrated financial markets can lead banks to specialise on a given market.

be compensated by a decrease in banks' total exposure, causing the overall systemic risk to decrease. This situation occurs, for example, when two banks switch from independent risky strategies (no common exposure, very high individual risk) to a common safe strategy (complete common exposure, but a very low risk).

In this paper, we try to clarify these theoretical ambiguities on the basis of an investigation using empirical data encompassing a panel of large international banks over a period running from 1993 through to 2006. Specifically, we pose the following questions: (1) what was the impact of international financial integration on banks' common exposure to shocks between 1993 and 2006? (2) What was the impact of international integration on systemic risk in the international banking sector between 1993 and 2006? (3) Is there an empirical link between common exposure to shocks and systemic risk during the period under consideration?

To estimate the impact of international integration on common exposure, we analyse the way in which the co-movement – i.e. correlation – of banks' asset-to-debt ratios (AD ratios henceforth) changes. AD ratios sum up banks' assets and liabilities as well as their interrelationship. Changes in the AD ratio can thus be considered a good summary of changes in the overall financial health of the banks. A high correlation between AD ratios suggests that both banks are similarly affected by shocks and, thus, that they have a high common exposure to shocks.

To estimate the impact of international integration on systemic risk, we compute a systemic risk index based on Lehar (2005), and study its changes during the period 1993-2006. Lehar's index measures the probability of observing a systemic crisis – defined as the simultaneous defaults of a given proportion of banks – in the banking sector at a given point in time. Finally, we investigate whether there is a link between movements in banks' common exposure (i.e. the correlation between AD ratios) and systemic risk.

Several studies have analysed correlations between variables relating to banks. De Nicolò and Kwast (2002) find a significant rise in stock return correlations between large US banking institutions during the 1990s. In a similar study for the European Union, Schröder and Schüler (2003) show that the correlations between 13 national bank stock indices have risen significantly in recent years. Brasili and Vulpes (2005) draw a similar conclusion when studying the correlations between distance-to-defaults for European banks. Hawkesby, Marsh, and Stevens (2007) analyse the correlations between equity returns (and between premiums for credit default swaps) of large and complex financial institutions (LCFI) in Europe and the US. The general conclusion from these studies is that co-movement between banks has increased in the last decade, which suggests that banks are increasingly exposed to the same risk factors. Most of these studies conjectured that the observed increase in co-movements leads to higher systemic risk.

Our paper differs from the previous studies in two ways: Firstly, we use a new method based on Ledoit, Santa-Clara, and Wolf (2003) to estimate the joint dynamic of the AD ratios as a whole (i.e. for all banks at the same time), whereas previous studies concentrated on the dynamic between pairs of banks. The resulting time-varying covariance matrix can then be used directly in the computation of the systemic risk index as well as for computing the evolution of correlation between banks. Secondly, and this is our main contribution, we study in detail the link between common exposure and systemic risk. We assess whether a higher common exposure to shocks (i.e. higher correlation) is associated with higher or lower systemic risk, or whether it plays no role in systemic risk. As mentioned before, other studies have often claimed that higher correlation yields to higher systemic risk, without formally verifying this assumption.

The main results of our analysis are the following. Firstly, we find that the correlation between banks' AD ratios decreases in the first part of the sample period, and increases after 2000. This suggests that, before 2000, banks specialised – and thus reduced their common exposure to shocks – rather than diversifying their portfolios in response to changes in the banking sector environment. After 2000, however, the banks appear to have become increasingly similar and their common exposure has risen. This finding holds for the whole sample as well as for different regional sub-groups (namely North America and European Union). However, the degree of common exposure differs between these groups. Correlations between North American banks tend to be higher than between their European counterparts. Co-movements between US and European banks are far less pronounced than within each regional sub-group, suggesting that these two groups are (at least partially) exposed to different shocks.

Secondly, as opposed to the correlation analysis, we do not find any significant trend in the systemic risk index. The latter is rather characterised by two peaks, one at the end of 1998 and the other at the end of 2002 and the beginning of 2003. These two periods correspond to two well-known episodes of high stress levels affecting the banking sector: the LTCM and Russian crisis at the end of 1998 and a persistent downturn on the stock market in 2002-2003. Taking a closer look at individual sub-groups, we find that the high level of systemic risk in 2002-2003 was mainly attributable to the fact that European banks were also suffering from the poor economic conditions in the European economy.

Thirdly, our results point out that correlation between banks is not a reliable measure of systemic risk. The link between correlation and our systemic risk index is weak and its direction can change, depending on the period considered. The distance-to-default, by contrast, which is a combination of the volatility and level of the AD ratio, turns out to be a very reliable explanatory factor with respect to the systemic risk index. In other words, systemic risk seems to be the consequence of each bank's individual risk taking (i.e. its distance-to-default), rather than of all banks' common exposure to shocks (correlation). This finding warns us against viewing systemic risk as a pure correlation phenomenon and highlights the danger of high and volatile leverage at the individual bank level. Note, however, that once the effects of the distance-to-default are taken into account, we find that correlation is positively associated with systemic risk. In other words, for a given level of individual risk, a higher common exposure implies a higher systemic risk.

The paper is structured as follows: Section 2 explains the methodology used to estimate the correlation dynamics and the systemic risk index. The data used is described in Section 3. Section 4 studies the correlation dynamics between large international banks' AD ratios. Section 5 presents the estimated systemic risk index and compares its evolution with the correlation dynamics. Section 6 gives our conclusions and recommendations for banking sector supervisors.

2 Methodology

As mentioned in the previous section, we need two ingredients to answer the questions at the centre of this paper: the evolution of the correlations between banks' AD ratios and the evolution of the systemic risk index. To get them, we proceed in four steps: (1) we make some assumptions on bank asset and debt dynamics, (2) we use these assumptions to recover the AD ratios from observable equity and debt data using Merton's method,[2] (3) we estimate the joint dynamic of the AD ratios, including the dynamic of their covariances, using a multivariate GARCH model and (4) we use the estimated dynamic to compute the systemic risk index. The next four sections describe these steps in details. In addition, the next section shows why shocks to AD ratios sum up both shocks to bank assets and debts as well as their effects on each others.

2.1 Asset and debt dynamic

The AD ratio is defined as the ratio between the asset market value of a bank and its debt face value. Unfortunately, asset market values are not directly

[2] The AD ratios are not directly observable because the asset market value is not directly observable.

observable, but, following Merton (1974), they can be estimated by modelling bank's equity as a call option on bank's assets. However, to recover the AD ratios from observed equity prices with Merton's technique, we have to assume that asset market values and debt face values follow a multivariate Itô process such as

$$\begin{bmatrix} d\mathbf{A}_t \\ d\mathbf{D}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A}_t & 0 \\ 0 & \mathbf{D}_t \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_D \end{bmatrix} dt + \begin{bmatrix} \mathbf{A}_t & 0 \\ 0 & \mathbf{D}_t \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}_{AA} & \boldsymbol{\Upsilon}_{AD} \\ \boldsymbol{\Upsilon}_{DA} & \boldsymbol{\Upsilon}_{DD} \end{bmatrix} \begin{bmatrix} d\mathbf{w}_A \\ d\mathbf{w}_D \end{bmatrix} \tag{1}$$

We have that $d\mathbf{A}_t$ and $d\mathbf{D}_t$ are $(N \times 1)$ vectors containing the instantaneous change in assets A_t^i and debts D_t^i of all banks, respectively. N is the number of banks. $\mathbf{A}_t$ and $\mathbf{D}_t$ are $(N \times N)$ matrices containing the assets A_t^i and the debts D_t^i in their respective diagonals, all other elements being zeros. $\boldsymbol{\mu}_A$ and $\boldsymbol{\mu}_D$ are $(N \times 1)$ vectors regrouping the constant instantaneous growth rate of banks' assets and debts, respectively. $d\mathbf{w}_A$ and $d\mathbf{w}_D$ $(N \times 1)$ are vectors of independent Wiener processes. They represent the individual shocks to assets and debts of each bank at time t. Finally $\boldsymbol{\Upsilon}_{AA}$, $\boldsymbol{\Upsilon}_{AB}$, $\boldsymbol{\Upsilon}_{BA}$ and $\boldsymbol{\Upsilon}_{BB}$ are $(N \times N)$ matrices which regroup the instantaneous responses of assets and debts to the different shocks. For example, v_{AD}^{ij} (the ij-th element of $\boldsymbol{\Upsilon}_{AD}$) is the instantaneous response of the bank i asset value to a shock in bank j debt.

The diagonals of the matrices $\boldsymbol{\Upsilon}_{AA}$ and $\boldsymbol{\Upsilon}_{DD}$ are the direct responses of the bank's assets and debt to their own shocks. The diagonal of the matrix $\boldsymbol{\Upsilon}_{AD}$ ($\boldsymbol{\Upsilon}_{DA}$) is the direct response of a bank's assets (debt) to a shock affecting its own debt (assets). All other elements are the indirect responses of a bank's assets and debt to shocks affecting other banks' assets and debt. They represent the contagion effects between banks through interlinkages. Note that we do not assume any symmetrical response between banks or between asset and debts. For example, the correlation between assets and debt can be different according to whether the shock affects the assets or the debt. Similarly, the response of bank i to a shock to bank j can be different from a bank j's response to bank i's shock.

Without loss of generality (see Appendix A), we can rewrite equation (1) in the reduced form

$$d\mathbf{z}_t = \boldsymbol{\mu} dt + \boldsymbol{\Upsilon} d\mathbf{w} \tag{2}$$

where

$$\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}' = \boldsymbol{\Omega}$$

$\mathbf{z}_t$ is the $(N \times 1)$ vector regrouping the log AD ratios $z_t^i = \ln(A_t^i / D_t^i)$, $\boldsymbol{\mu}$ is a $(N \times 1)$ vector of instantaneous drifts in log AD ratios and $\boldsymbol{\Omega}$ is the $(N \times N)$ variance-covariance matrix between instantaneous changes in log AD ratios.

As shown in Appendix A, the matrix $\boldsymbol{\Upsilon}$ sums up all interactions between banks' assets and debts.

2.2 Recovering AD ratios from equity prices

Merton (1974) suggested to model bank's equity as a call option on bank's assets to compute the bank's default probability estimated by market participants. This method can also be used to recover the bank's AD ratio from the equity price. Merton's method is based on the fact that if, at debt's maturity time $t+T$, the value of the bank's assets is smaller than its debt ($A^i_{t+T} < D^i_{t+T}$), then it is not rational for the shareholders to exercise the option, i.e., they will make the bank default. If the bank defaults, then the value of the equity is zero. Thus Merton's model states that the value of bank i's equity at time $t+T$ is:

$$E^i_{t+T} = \max\left(A^i_{t+T} - D^i_{t+T}, 0\right) \tag{3}$$

where E^i_t is the bank's stock price.

An equity with such payoffs is similar to an exchange option[3]. If both the assets and the debt are log normally distributed, as stated in Equation (1), then the option initial value can easily be computed and is equal to (Margrabe, 1978):

$$E^i_t = A^i_t \Phi\left(d_1\right) - D^i_t \Phi\left(d_2\right) \tag{4}$$

with

$$d_1 = \frac{z^i_t + \left(\sigma^2_{z^i_t}/2\right) T}{\sigma_{z^i_t}\sqrt{T}}$$

$$d_2 = d_1 - \sigma_{z^i_t}\sqrt{T}$$

where $\Phi\left(\cdot\right)$ is the cumulative normal distribution and $\sigma^2_{z^i_t}$ is the conditional variance of the log AD ratio. Dividing both sides of Equation (4) by D^i_t yields

$$X^i_t = Z^i_t \Phi\left(d_1\right) - \Phi\left(d_2\right) \tag{5}$$

where X^i_t is the equity-to-debt ratio of bank i at time t and Z^i_t is its AD ratio at time t. Using Itô's lemma, we have that

$$\sigma_{x^i_t} X^i_t = \sigma_{z^i_t} Z^i_t \Phi\left(d_1\right) \tag{6}$$

where $\sigma^2_{x^i_t}$ is the conditional variance of the log equity-to-debt ratio.

Bank i's debt D^i_t and equity price E^i_t are directly observable. With them, we

[3] Exchange options are sometimes also referred to as options to exchange one asset for another.

can form the equity-to-debt ratio X_t^i and compute its conditional variance $\sigma^2_{x_t^i}$. With X_t^i and $\sigma_{x_t^i}$ known, we are left with two unknown variables in Equation (5) and (6): the AD ratio Z_t^i and its conditional variance $\sigma_{z_t^i}$. The AD ratio Z_t^i can be recovered by simultaneously solving Equation (5) and (6) using a numerical iterative process.[4,5]

Note that this method gives the correct AD ratio only if the market participants correctly interpret the information they have about the banks. In particular, they should understand correctly the inter-dependence between banks and integrate it in their valuation. Note also that this method is not the only one that can be used to recover the AD ratio.[6] However, Hovakimian and Kane (2000) show that the differences in the AD ratio valuations given by different methods are small.

2.3 Estimation of the AD ratio joint dynamics

The next step is to estimate the dynamic process in Equation (2) with empirical data. We choose to model the AD ratio dynamic with a multivariate GARCH model (M-GARCH). The main advantage of this kind of model is that it allows for *time-varying* variances and covariances. This is necessary in our context since we are interested in the evolution of the correlation through time.

The equivalent of Equation (2) in discrete time is:

$$\begin{aligned} \Delta \mathbf{z}_t &= \boldsymbol{\mu}_z + \mathbf{u}_t \\ \mathbf{u}_t &\sim N\left(0, \mathbf{H}_t\right) \end{aligned}$$

where $\Delta \mathbf{z}_t$ is a $(N \times 1)$ vector containing the changes in log AD ratios Δz_t^i, $\boldsymbol{\mu}_z$ is a $(N \times 1)$ vector of constants, $\mathbf{u}_t$ is a $(N \times 1)$ vector of white noise residuals and $\mathbf{H}_t$ denotes the $(N \times N)$ conditional variance-covariance matrix of the

[4] Note that only the "marginal" variance of the equity-to-debt ratios is necessary to recover the AD ratios. In particular, this technique does not require to know the correlations between the banks' equities (see Zhou, 2001, for another illustration with a bivariate model). Therefore, this technique can be applied separately to each bank.

[5] Other techniques are available to recover the market asset value. Duan (1994) estimates it by maximizing a likelihood function. Vassalou and Xing (2004) use an iterative process that does not require equation (6).

[6] See for example, Bichsel and Blum (2004) for a model without stochastic debt, Ronn and Verma (1986) for a model with forbearance or Saunders and Wilson (1995) for a model with an infinite-maturity put.

residuals. We use the diagonal *Vech* model (Bollerslev, Engle, and Wooldridge, 1988) specification for the dynamic of $\mathbf{H}_t$. In this model, the conditional covariance h_t^{ij} between bank i and bank j depends only on the past covariance and the past residuals:

$$h_t^{ij} = \omega_{ij} + \alpha_{ij} u_{t-1}^i u_{t-1}^j + \beta_{ij} h_{t-1}^{ij} \tag{7}$$

The model has the property that when two banks experience trouble at the same time, their correlation increases and they will tend to move together in the future. This property is important as it captures the empirical finding that the correlation between banks seems to increase during bad periods. Yet, correlation will also increase when both banks are hit by a positive shock; i.e., the covariances behave symetrically.

In matrix form, Equation (7) has the following representation

$$\mathbf{H}_t = \mathbf{C} + \mathbf{A} \otimes \mathbf{u}_{t-1}\mathbf{u}_{t-1}' + \mathbf{B} \otimes \mathbf{H}_{t-1} \tag{8}$$

where the coefficient matrices $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ are $(N \times N)$ matrices regrouping the parameters ω_{ij}, α_{ij} and β_{ij}, respectively. The symbol $\otimes$ denotes the Hadamar product of two matrices.[7] The covariance matrix $\mathbf{H}_t$ has some distinctive characteristics to be respected by the estimation method. First, $\mathbf{H}_t$ is symmetric implying that $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ must also be symmetric. Secondly, $\mathbf{H}_t$ is a positive semidefinite matrix and thus, any estimation of it must also be positive semidefinite.

A natural way to estimate $\mathbf{H}_t$ and the coefficient matrices $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ seems to use maximum likelihood estimates as it is usually done for univariate GARCH models. Unfortunately, this is not feasible because (i) the parameters are too numerous and so intricately linked that existing optimization algorithms do usually not converge, and (ii) maximum likelihood estimation does not necessarily give positive semidefinite covariance matrices. To cope with this second problem, econometricians usually impose additional conditions on the model coefficients to ensure that the matrix $\mathbf{H}_t$ is positive semidefinite.[8] In addition to the fact that such restrictions might not make sense from an economic point of view, Kroner and Ng (1998) have shown that M-GARCH results are very sensitive to different specifications.

We choose to follow a different approach to estimate the coefficient matrices $\mathbf{C}$,

[7] The Hadamar product is the elementwise product of two matrices: $\mathbf{U} \otimes \mathbf{V} = (u_{ij}v_{ij})$.

[8] See Ding and Engle (2001) for a recent comparison of the restrictions used by different models.

A and **B**: the Flexible M-GARCH method developed by Ledoit, Santa-Clara, and Wolf (2003). This procedure has the advantage to solve both problems previously mentioned without imposing *a priori* restrictions on the coefficients. It is based on a *decentralized* estimation of the coefficients. Ledoit *et al.* propose to estimate the coefficient matrices **C**, **A** and **B** in two steps. In the first step, each coefficient of the matrix is independently estimated with a univariate or bivariate GARCH model. Thus, the estimation of a large matrix is reduced to several univariate and bivariate problems for which conventional univariate and bivariate GARCH estimation techniques are easy to apply. As indicated above, the resulting estimated coefficient matrices $\hat{\mathbf{C}}$, $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ will not necessarily ensure the variance-covariance matrix to be positive semidefinite. Thus, in a second step, Ledoit *et al.* apply a result from Ding and Engle (2001) stating that positive semidefinite coefficient matrices are a sufficient condition to yield (almost surely) a positive semidefinite variance-covariance matrix. The second part of the procedure is thus to find the positive semidefinite matrices $\tilde{\mathbf{C}}$, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ that are the closest to the $\hat{\mathbf{C}}$, $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ initial matrices.[9] The coefficient matrices $\tilde{\mathbf{C}}$, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ given by the Flexible M-GARCH estimation are then used to compute the conditional variance-covariance dynamic with Equation (7) and, in the next step, to compute the systemic risk index.

2.4 Construction of the systemic risk index

Our systemic risk index follows Lehar (2005). The index is an estimation of the *probability of a systemic banking crisis* at time t. In this paper, a systemic crisis is defined as follows:

Definition 1 *A systemic crisis occurs when a percentage θ of the banking system becomes insolvent within the next k periods.*

Definition 1 requires another definition, which specifies under which conditions a bank defaults.

Definition 2 *The bank i defaults if the market value A_t^i of its assets falls below the face value D_t^i of its debt within the next k periods (i.e. $A_{t+j}^i < D_{t+j}^i$ for at least one $j \in [0,k]$ or equivalently $Z_{t+j}^i < 1$ for at least one $j \in [0,k]$).*

Given Definitions 1 and 2, the systemic risk index can be expressed as:

$$I_t(\theta) = \Pr\left[\sum_{i=1}^{N} \theta_t^i b_t^i > \theta\right] \tag{9}$$

[9] The positive semidefinite matrice $\mathbf{X}$, which is the closest from the initial matrice $\mathbf{Y}$, can be found by using a simple algorithm developped by Sharapov (1997).

where N is the number of banks, θ_t^i is the weight[10] of bank i in the banking system at time t and

$$b_t^i = \begin{cases} 1 \text{ if } A_{t+j}^i < D_{t+j}^i & \text{for at least one } j \in [0, k] \\ 0 & \text{otherwise} \end{cases}$$

b_t^i is a dummy variable that takes the value 1 if bank i goes bankrupt in the next k periods. There are various ways to determine a bank's weight θ_t^i in the banking sector (e.g. equal weight, proportion of a bank's assets in the total banking sector's assets, proportion of a bank's interbank deposits in the total system, etc.). We chose to give an equal weight to each bank.

Unfortunately, it is not possible to compute analytically the probability of a systemic crisis. To estimate it, we proceed with a Monte-Carlo simulation based on the estimated AD ratio joint dynamics described in section 2.3. An alternative would have been to use approximation techniques such as the one proposed by Carmona and Durrleman (2006). The Monte-Carlo simulation, yet, has the advantage of being easier and faster to implement and more flexible for further developments (e.g. modification of the dynamic or explicit modelling of contagion effects). A description of the simulation algorithm to estimate the systemic risk index is given in Appendix B. We set $k = 12$ (one year) in this paper.

3 Data

The data needed to compute the correlation between AD ratios and to construct the systemic risk index consist of individual banks' balance sheet data (debt) and market information (equity prices). Data on debt is taken from Bloomberg while equity prices stem from Datastream. As data on debt is not available on a monthly basis, quarterly and – for some banks – yearly data have been transformed into monthly data by linear interpolation.

We constructed two different datasets. The first dataset comprises monthly data on 27 large international banks from November 1992 until June 2006 (long sample).[11] The second dataset (short sample) comprises data on a total of 39 large international banks – including the 27 institutions already represented in the first dataset – from June 1997 until June 2006.[12]

[10] The individual weights are normalized such that $\sum_{i=1}^{N} \theta_i = 1$.
[11] The long sample consists of banks from: Germany (3), France (1), Italy (2), Netherlands (2), Spain (2), Sweden (2), Switzerland (1), UK (3), USA (2), Canada (5), Australia (4).
[12] The short sample consists of banks from: Belgium (3), Germany (3), France (2),

4 Common exposure to shocks

To get an idea of how banks' common exposure to shocks has evolved in time, we try to identify a potential common trend in AD ratio correlations between pairs of banks. A high AD ratio correlation indicates that two banks are both equally affected by a shock, i.e., that they have a high common exposure to shocks. Thus, if we observe a common upward trend in all correlations, we can conclude that, in the aggregate, banks' common exposure has increased and that banks have become more similar.

We can get a first idea on the trend in correlation by observing the evolution of the average correlation (cf Figure 1). In both samples, the average correlation decreases until about 2000 and then increases regularly. This pattern is also observed in the average correlation between pairs of North American banks, pairs of European Union banks and between pairs of banks from each sub-group. This is a first indication that, in the aggregate, banks' common exposure to shocks has decreased until about 2000 and then increased.

To study more precisely this hypothesis, we estimate the common trend to all correlations in a panel data analysis. We then test for a break in the slope of the common trend. Concretely, we estimated the following system of equations

$$y_t^{ij} = \gamma + \delta t + u^{ij} + \varepsilon_t^{ij} \tag{10}$$

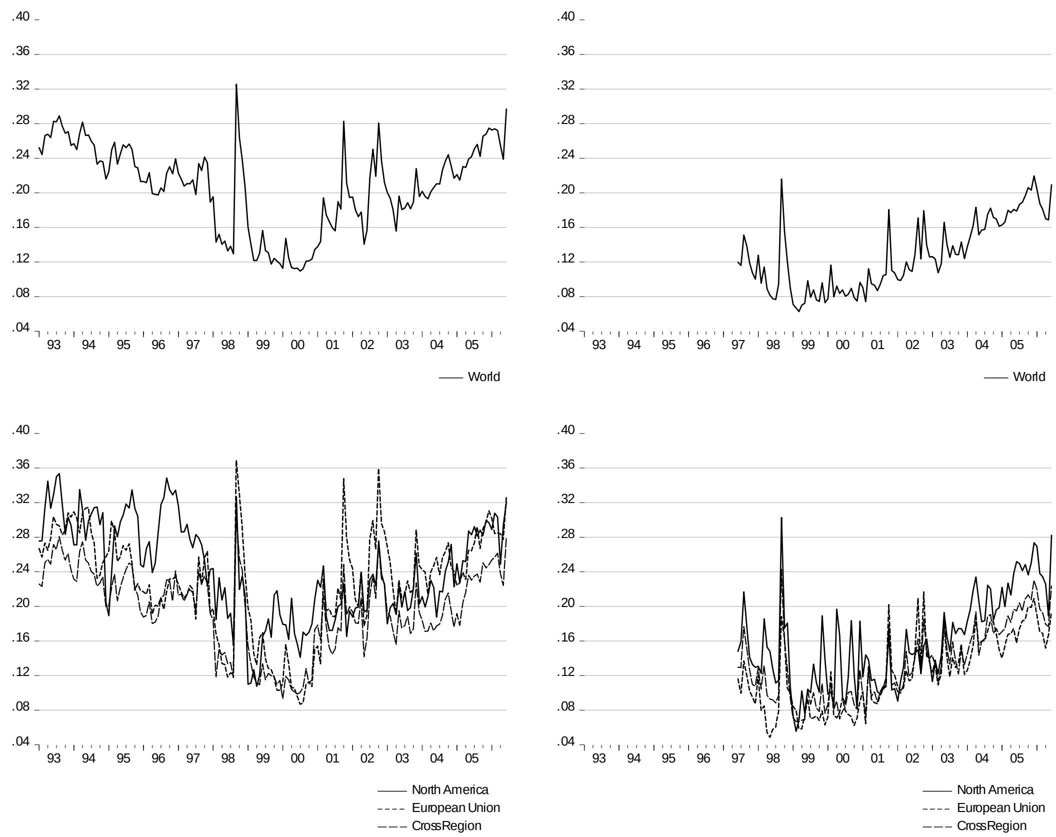
where y_t^{ij} is a logit transformation[13] of the correlation between bank i and j at time t, γ is the average (logit) correlation, δ is the slope of the common trend, u^{ij} is a fixed effect particular to each pair and ε_t^{ij} is a independant heteroskedastic error term, which is normally distributed with variance σ_{ij}^2. We then test the hypothesis of a break at time t_0 in the slope δ of the trend. We used the test developed by Bai and Perron (1998, 2003) which simultaneously estimates the most probable break date and then tests if the break is statistically significant. The results are presented in Table 1.

The estimated break dates lie between August 1999 and June 2000 (with an exception of a break date of November 2001 for the North American banks in the short sample). It indicates that a change in banks' common exposure occurred around the beginning of this century. Before this date, the trends are negative, which implies that the common exposure to shocks had a tendency

Italy (3), Netherlands (2), Spain (2), Sweden (2), Switzerland (1), UK (5), USA (7), Canada (5), Australia (4).

[13] More precisely, $y_t^{ij} = \ln\left(q_t^{ij}/\left(1-q_t^{ij}\right)\right)$ where $q_t^{ij} = \left(cor\left(i,j\right)+1\right)/2$. This transformation insure that y_t^{ij} is distributed over $]-\infty;+\infty[$ whereas the correlation is bounded between -1 and 1.

Fig. 1. Correlation between AD ratios (left: long sample, right: short sample)



to decrease. This suggests that, during this period, banks have rather chosen to specialize than to diversify their portofolios. After 2000, the trends reverse and common exposure to shocks increases, hinting at increasing similarities or interdependencies between banks. An increase in banks' co-movements since 1999 is also documented by Brasili and Vulpes (2005).

This trend reverse is observed in both regions and between these regions. Figure 2 displays the estimated trend for correlation between North American banks, between EU banks and between banks of each region. [14] Except for the end of the sample, the correlation between North American banks is higher than between EU banks. North American banks seem to be more commonly exposed to shocks, or more homogeneous, than EU banks. The correlation between EU and North American banks is the lowest, indicating that banks from different regions are less commonly exposed to shocks or more heterogeneous. This result is in line with Hawkesby, Marsh, and Stevens (2007), who find a high degree of heterogeneity between both sub-groups and a higher correlation between US banks. Note finally that the slope of the EU banks is steeper

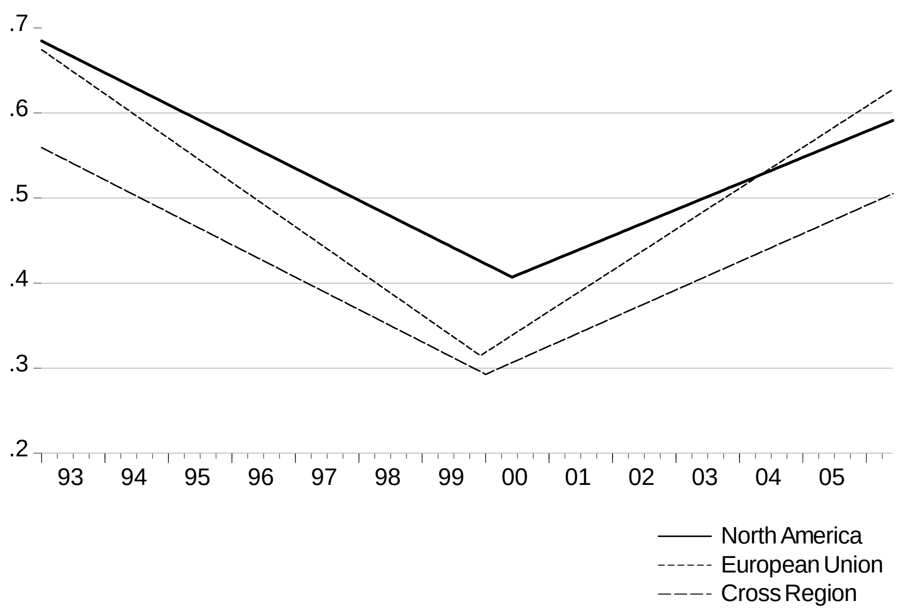
[14] The trend for the short sample are not presented here but can be obtained by the authors. The conclusions are similar.

Table 1
Test for a break in the common trend and estimated slopes of the trend

	Region	sup$F$ -stat	p-value	Break date date	γ	δ before break	δ after break
1993-2006	World	1604.05	<1%	2000.04	0.5918	-0.0034	0.0033
	EU	491.46	<1%	1999.12	0.6742	-0.0043	0.0040
	North America	100.82	<1%	2000.06	0.6845	-0.0031	0.0026
	Cross	362.65	<1%	2000.01	0.5593	-0.0032	0.0028
1997-2006	World	644.99	<1%	1999.10	0.2488	-0.0026	0.0027
	EU	114.28	<1%	1999.08	0.2308	-0.0025	0.0027
	North America	199.37	<1%	2001.11	0.3084	-0.0011	0.0049
	Cross	276.64	<1%	1999.12	0.2698	-0.0028	0.0030

All coefficients are signficant at a 1% confidence level. A supF-statistics greater than 16.64 indicates that the null hypothesis of no break is rejected at a 1% confidence level.

Fig. 2. Estimated common trends in AD ratios correlation (long sample, in logit tranformed units)



in both phases. EU banks have specialized more strongly during the pre-2000 period and then have become more similar than North American banks, to the point that the correlation between EU banks seems higher now than between US banks.

5 Evolution of systemic risk

Do the changes observed in the banking industry in the past years have any impact on systemic risk? In particular, does the increase in common exposure to shocks observed since 2000 generate a higher systemic risk? To answer these questions, we constructed a set of indices of systemic risk. These indices are based on Lehar (2005). They reflect the probability to observe a systemic crisis in the banking sector (cf. Section 2.4). We computed a global systemic risk index for all the banks in our sample and two regional sub-indices, one for North American banks and one for EU banks. For all three regions (World, North America, EU), we computed two indices: one for which a default of 10% of the banking sector triggers a crisis, one for which 20% of the banking sector must default to trigger a crisis. We computed these 6 indices for both samples (long and short samples), which makes a total of 12 systemic risk indices. These indices are presented in Figure 3.

The global index points out two periods of high systemic risk: at the end of 1998 and at end 2002 until beginning 2003. These two episodes correspond to the LTCM and Russian crisis in 1998 and to the stock market downturn in 2002-2003. The systemic risk during the rest of the sample is less acute. The 1998 peak is observed in both the US and the EU sub-indexes. The EU banks seem to have been more affected than the North American banks in 2002-2003, probably because they were also facing bad economic conditions at that time. In the US banking system, a period of higher systemic risk is also observed in 1994-1995, which translates by a slightly higher systemic risk in the global index.

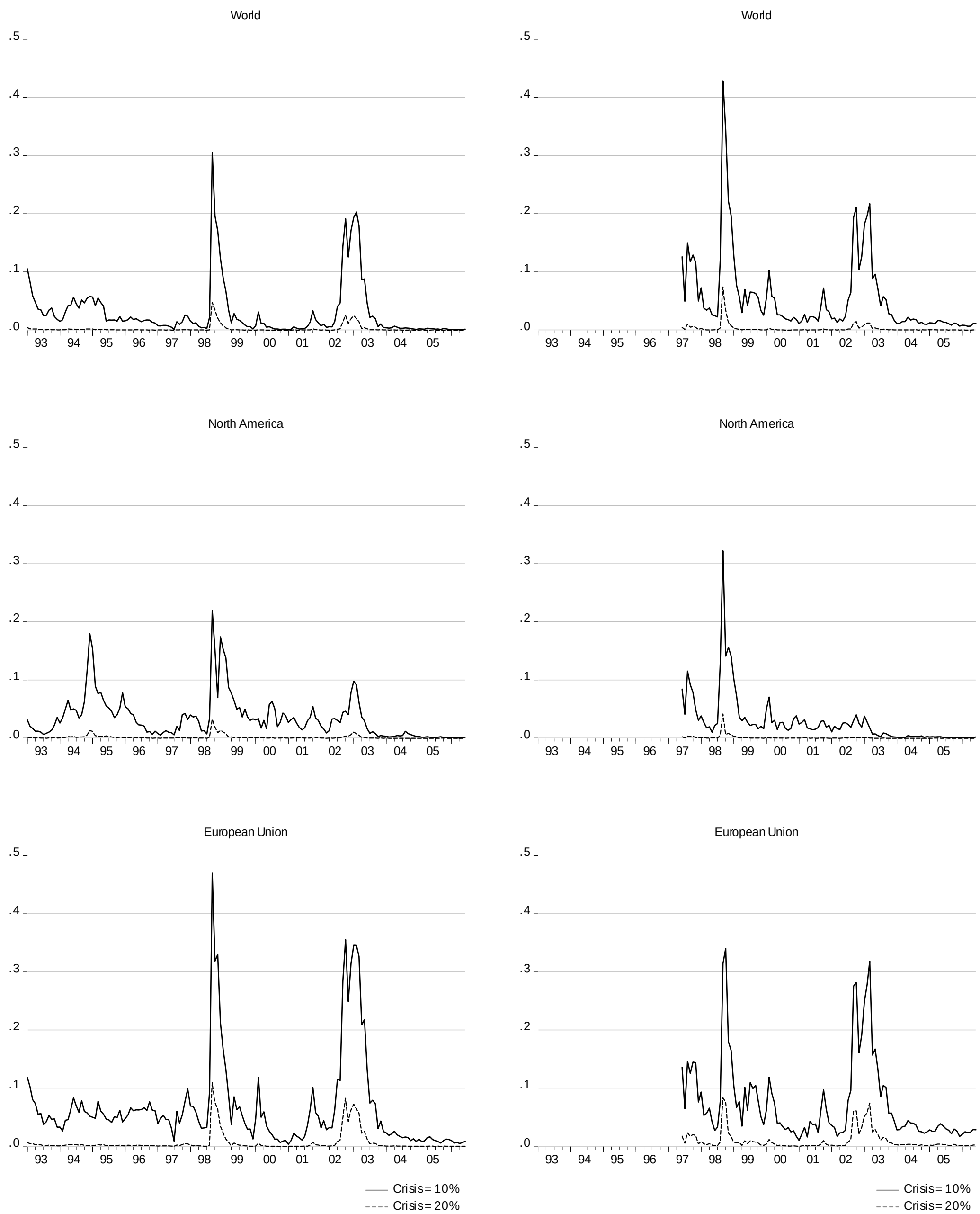
5.1 Trends in the systemic risk index

A quick look at Figure 3 suggests that the path of the systemic risk index is very different from the evolution of banks' common exposure to shocks presented in Figure 1. The latter has a distinct V-shape, whereas the former is characterized by two peaks of higher systemic risk for the banking sector. This visual impression is confirmed when we try to fit a trend with a break at the beginning of 2000 (which corresponds to the break date observed in the correlation trend) to the systemic risk index. The results of this regression are presented in Table 2. Since the index is a probability bounded between 0 and 1, we estimate the coefficients with a logit regression using weighted least squares as suggested by Greene (2000).[15]

Most indices do not display any significant trend. Only the global and the

[15] All regressions in this section are made using this method.

Fig. 3. Systemic risk index (left: long sample, right: short sample)



North American systemic risk indices seem to have significantly decreased before 2000 in the long sample and after 2000 in the short sample, respectively. Furthermore, no significant break date is detected by the Bai and Perron test for any of the indices. This result contrasts with the unambiguous trends and breaks observed in the AD ratio correlation: While a clear V-shaped trend appears in the dynamic of banks' common exposure to shocks, no apparent trend is detected in the systemic risk index' pattern . Moreover, the slope of

Table 2
Trend in the systemic risk index

	Index	Trend before 2000	Trend after 2000
1993-2006	World 10%	-0.0196**	0.0109
	World 20%	-0.0233**	0.0034
	North America 10%	0.0075	-0.0110
	North America 20%	0.0018	-0.0079
	EU 10%	0.0031	0.0019
	EU 20%	0.0022	0.0142
1993-2006	World 10%	-0.0151	-0.0076
	World 20%	-0.0435	0.0012
	North America 10%	-0.0223	-0.0343**
	North America 20%	-0.0397	-0.0118**
	EU 10%	-0.0120	-0.0030
	EU 20%	-0.0299	-0.0004

* (**) indicates that the coefficient is significant at a 5% (1%) confidence level. The percentage associated to each index corresponds to the proportion of the banking system that need to default to trigger a crisis (e.g.World 10% corresponds to the systemic risk index for which a 10% of the banking system has to default to trigger a crisis).

the significant trend observed after 2000 for the North American systemic risk index does not correspond to the sign that one would *a priori* expect (i.e. the systemic risk decreases whereas the common exposure to shocks increases).

Many other studies record similar results as ours for banks' common exposure to shocks or banks' co-movements. Most of them conclude, without explicitly checking it, that an increase in co-movements induces a higher systemic risk. However, given our results for the systemic risk index, the existence of the link between co-movements or common exposure to shocks and systemic risk is ambiguous. The next section studies this question in more details.

5.2 Are common exposure and systemic risk related?

The results of the previous section raise questions about the existence of a link between banks' common exposure to shock (i.e. AD ratio correlations) and systemic risk in the banking sector. Do common exposures really play a role for systemic risk? How can we interpret a change in common exposure in terms of systemic risk? Considering the construction of the systemic risk index (see Section 2.4), it is obvious that three elements determine its value: 1) the *correlation structure* between banks' AD ratios, 2) the *volatilities* of the AD ratios and 3) the *level* of the AD ratios. While the first component captures the systemic characteristics of a banking sector, the last two components are bank-specific. Combined in the *distance-to-default*[16], they describe bank's individual risk taking. The systemic risk index is a function of these systemic and bank-specific dimensions. Unfortunately, we do not know the exact form of this function. We can guess though that it is likely to be non linear.

To get an idea about each factor's influence on the systemic risk index, we compute the rank correlation between the systemic risk index and (i) the banks' AD ratio correlations and (ii) the banks' distance-to-default. The rank correlation statistics are preferred to the traditional (linear) correlation (Pearson coefficient) because they measure the link between two variables independently of the form taken by the function linking them. We use both the Spearman rank-order correlation coefficient and the Kendall measure of correlation to compute the rank correlation.[17] We start by computing the rank correlation between the *average* correlation (or, alternatively, between the average distance-to-default) and the systemic risk index. Note, however, that it is difficult to adequately reflect the complete correlation structure (or the complete distance-to-default structure) in one single measure such as the average. In particular, it is possible that the systemic risk index might be mainly influenced by extreme values of correlations or distance-to-defaults (i.e., by banks that are extremely commonly exposed or extremely close to default). To check for that, we also use the 75% (25%) and the 90% (10%) percentiles of the correlations (distance-to-defaults). The evolution of the average and centiles of correlations and distance-to-defaults are displayed in Figure 4. The rank correlation between the systemix risk index and these different measures are presented in Table 3.[18]

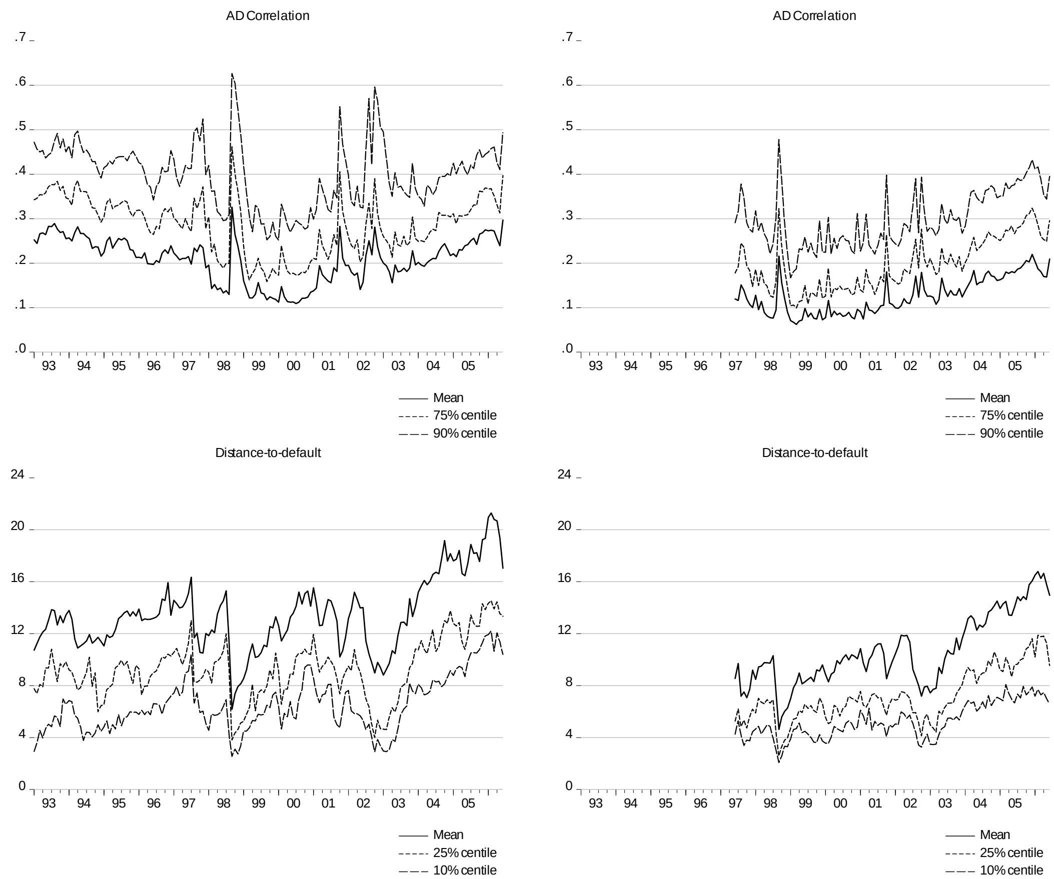
The results for the rank correlation show that the link between systemic risk and banks' common exposure (i.e. AD ratio correlation) is ambiguous. A positive relationship is identified in the long sample, while the same relationship appears to be negative in the short sample. On the opposite, the link between

[16] The distance-to-default is equal to the level of the AD ratio divided by its volatility.

[17] Spearman rank-order correlation coefficient measures the *linear* correlation between the *ranks* of each observation. Kendall's τ is even more nonparametric since it uses the relative ordering of the data, wihtout assuming any linear relation at any point of its computation.

[18] We present the results for the World index only. The results for the regional systemic risk indices can be obtained by the authors. The conclusions do not differ from those presented here.

Fig. 4. Evolution of the AD correlations and the distance-to-defaults (left: long sample, right: short sample)



systemic risk and distance-to-default is always negative. Moreover, the rank correlation between systemic risk and distance-to-default is always stronger than the one between systemic risk and common exposure. We thus draw the following main conclusion: Low distance-to-default is a much stronger and much more reliable sign of high systemic risk than high correlation. The effect of banks' common exposure to shocks on systemic risk is weaker and can even change direction depending on the period.

The strong link between distance-to-default and systemic risk is illustrated by Figure 5, in which a logit transformation of the systemic risk index is plotted against AD ratio correlation (left) and distance-to-default (right), respectively. Clearly, the dispersion with the distance-to-default is smaller than with correlation. Interestingly, with this transformation, the link between the systemic risk and the distance-to-default seems to be relatively linear.

Table 4 presents the results of a linear regression of the (logit of the) systemic risk index on the AD ratio correlation and on the distance-to-default, respec-

Table 3
Rank correlation between the systemic risk index and different factors

	Index	Factor	Correlation		Distance-to-default	
			Spearman coefficient	Kendall's tau	Spearman coefficient	Kendall's tau
1993-2006	World 10%	Average	0.2369	0.1620	-0.8587	-0.6833
		75% percentile	0.2874	0.1949	-0.8549	-0.6752
		90% percentile	0.4291	0.2880	-0.9046	-0.7429
	World 20%	Average	0.3041	0.2254	-0.7880	-0.6379
		75% percentile	0.3310	0.2464	-0.7553	-0.6093
		90% percentile	0.4731	0.3518	-0.8008	-0.6470
1997-2006	World 10%	Average	-0.3510	-0.2339	-0.9348	-0.7919
		75% percentile	-0.3424	-0.2319	-0.9375	-0.7976
		90% percentile	-0.3041	-0.2159	-0.8984	-0.7256
	World 20%	Average	-0.0618	-0.0472	-0.7630	-0.5939
		75% percentile	-0.0579	-0.0429	-0.7730	-0.6097
		90% percentile	-0.0470	-0.0401	-0.6837	-0.5238

The fourth (fifth) column gives the Spearman (Kendall) coefficient between different systemic risk indices and the mean, 75% centile and 90% centile of the correlation between banks' AD ratio. The sixth (seventh) column gives the Spearman (Kendall) coefficient between different systemic risk indices and the mean, 25% centile and 10% centile of the banks' distance to default.

tively. The coefficient on the distance-to-default is significantly negative in all specifications. The degree of correspondance (coefficient of partial correlation R^2) between the index and the estimated regression is very high (mostly over 80%). The results from the regression with correlation are less convincing: We have a very low coefficients of partial correlation (with the exception of the 90% centile in the long sample) and in the short sample, most coefficients are not significant.

Not surprisingly, these regression results coincide with those obtained from the rank correlation analysis. The common exposure (i.e. AD ratio correlation) is a poor predictor of the systemic risk index and the direction of its relation changes depending on the period. The distance-to-default, by contrast, explains well the systemic risk index. This difference in precision is illustrated by Figure 6. In each panel, the actual systemic risk index is compared with its value estimated with a linear function of one of the two factors (i.e. common exposure and distance-to-default, respectively). The match between the systemic risk index and its estimation based on the 10% percentile

Table 4
Regression of the systemic risk index on correlation and distance-to-default

		World 10%		World 20%	
	Regressor	Coefficient	R^2	Coefficient	R^2
1993-2006	Mean correlation	7.6496*	0.0396	17.4344**	0.1416
	75% centile of correlations	10.4503**	0.1271	20.1928**	0.3220
	90% centile of correlations	15.8976**	0.5599	25.3561**	0.7423
	Mean distance-to-default	-0.5073**	0.7761	-0.8087**	0.8164
	25% cent. of distance-to-defaults	-0.6618**	0.8601	-1.1114**	0.8683
	10% cent. of distance-to-defaults	-0.9528**	0.8545	-1.4776**	0.7810
1997-2006	Mean correlation	-4.1073	0.0234	-0.4085	0.0001
	75% centile of correlations	-3.2653	0.0306	-0.3175	0.0001
	90% centile of correlations	-3.6337*	0.0473	0.0664	0.0000
	Mean distance-to-default	-0.5672**	0.8735	-0.9852**	0.8830
	25% cent. of distance-to-defaults	-0.7476**	0.8908	-1.2656**	0.8935
	10% cent. of distance-to-defaults	-1.0445**	0.7599	-1.9254**	0.7650

* (**) indicates that the coefficient is significant at a 5% (1%) level.

of distance-to-default is striking (right and lower panel of Figure 6). Taking distance-to-default as a proxy for measuring systemic risk seems to give a fairly good and easily obtainable estimation of it.

However, while the distance-to-default seems to be the main factor driving the systemic risk, the common exposure might account for the portion of the systemic risk index that is not explained by the distance-to-default. To check that, we compute the rank correlation between the common exposure and the residuals of a regression of the systemic risk index on the distance-to-default. The idea is to check if a positive residual (i.e., an "excess" of systemic risk given what is estimated by distance-to-default) is associated with a high or a low common exposure. The rank correlations are presented in Table 5.

The results show that residuals are positively correlated with common exposure. For example, the Spearman rank correlation between the mean AD ratio correlation and the systemic risk left unexplained by the mean distance-to-default (residuals of the regression of the systemic risk on the mean distance-to-default) is 0.65. This degree of correlation is significantly higher than between common exposures and the systemic risk index (cf. Table 3), suggesting

Fig. 5. Systemic risk index vs. AD correlations (left) or distance-to-default (right) (long sample)

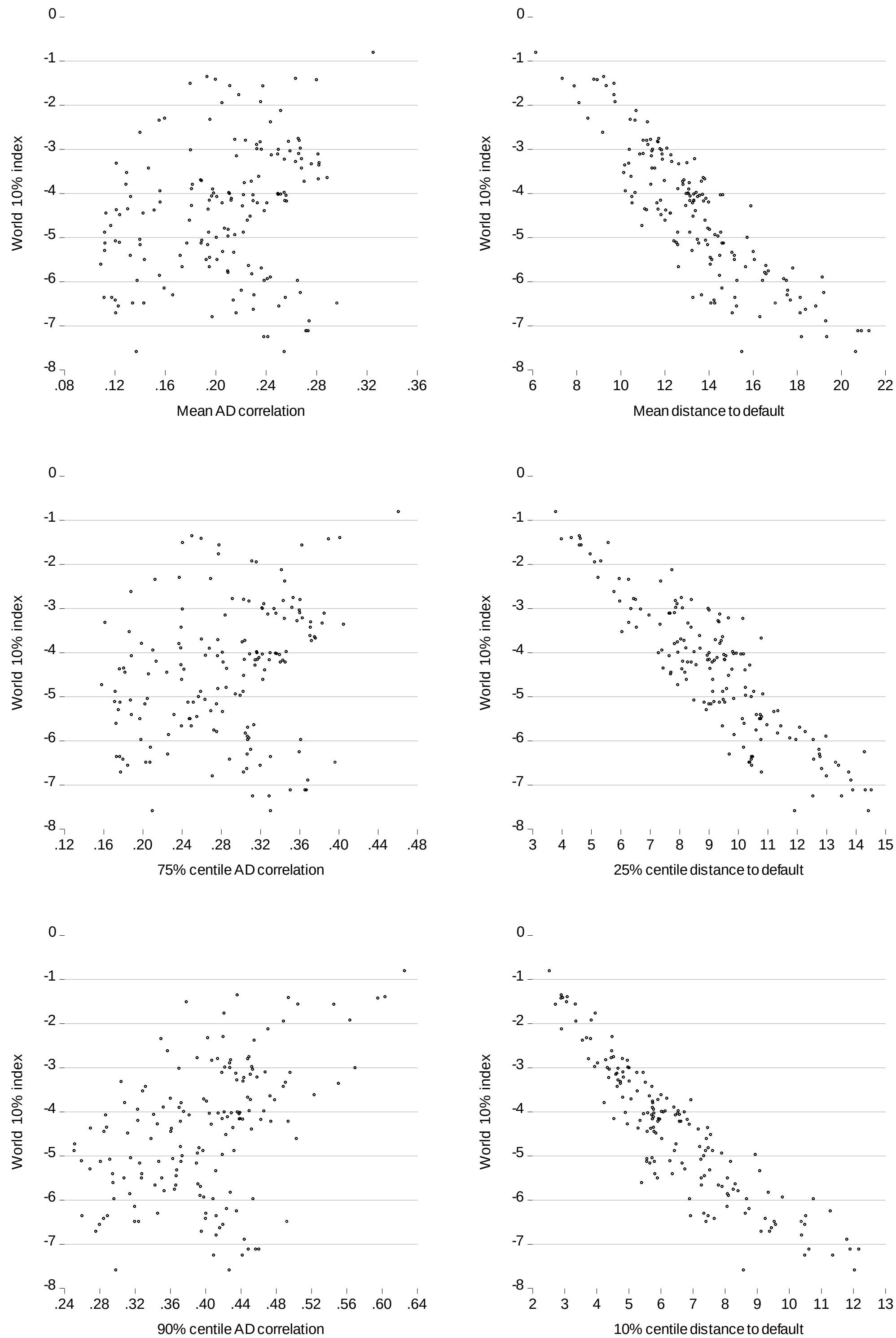
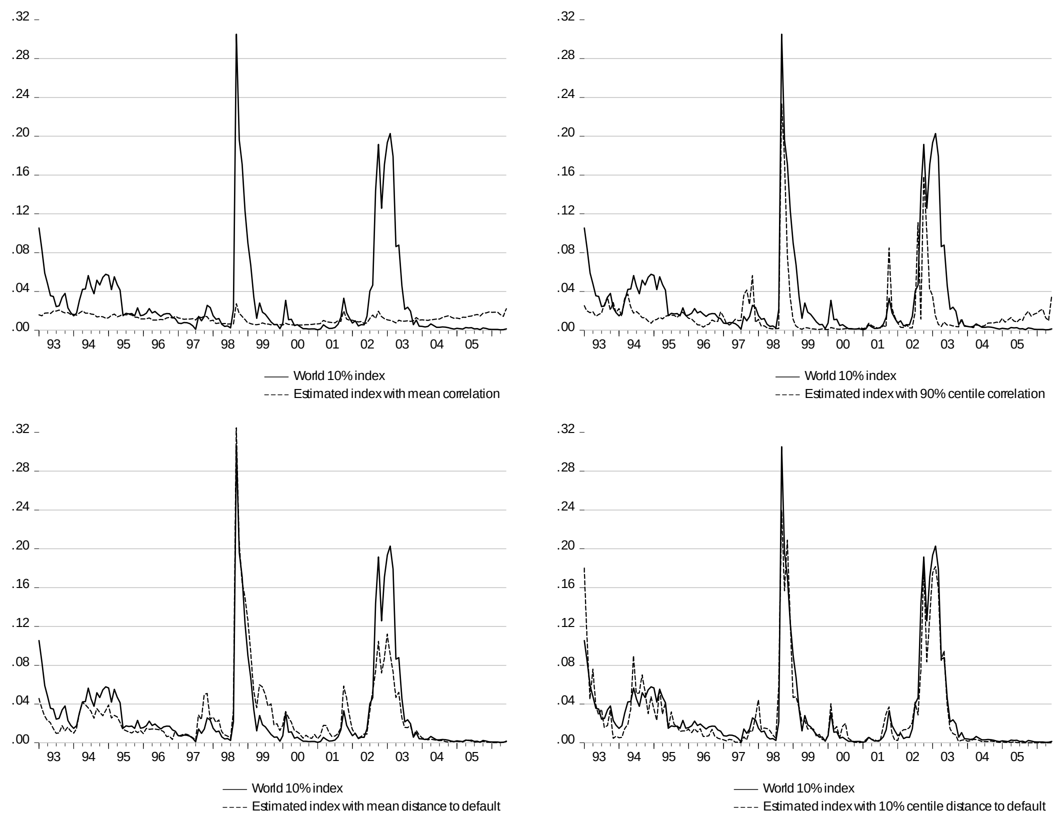


Fig. 6. Estimated systemic risk index (long sample)



that the effect of common exposures on the unexplained part of systemic risk is greater than on the systemic risk itself. We also see that the rank correlations are always positive. This means that, *once the distance-to-default is taken into account*, a higher common exposure always induces a higher systemic risk.

To summarize, we find that the systemic risk dynamic does not match the dynamic observed for banks' common exposure. This indicates that common exposure is probably not the main factor explaining systemic risk. Indeed, further analyses reveal that the banks' distance-to-defaults, which describes banks' individual risk taking, is the main force driving systemic risk. However, we find that common exposures explain relatively well the part of systemic risk left "unexplained" by the distance-to-default. We also find that, once the distance to default is taken into account, higher common exposure induces higher systemic risk.

Note that we have also tried to disentangle the effect of the distance-to-default between the AD ratios level, which represents the reserves that the banks can use to absorb shocks, and its volatility, which measures the risk of their investments. We found that both elements are of equal importance to explain the evolution of the systemic risk. The volatility plays a significant role in

Table 5
Rank correlation with residuals

	Residuals from equation using:	Factor	Spearman coefficient	Kendall's tau
1993-2006	Mean distance-to-default	Mean correlation	0.6468	0.4638
		75% perc. correlation	0.5845	0.4142
		90% perc. correlation	0.5158	0.3677
	10% perc. distance-to-default	Mean correlation	0.3944	0.2752
		75% perc. correlation	0.3476	0.2366
		90% perc. correlation	0.3079	0.2079
1997-2006	Mean distance-to-default	Mean correlation	0.6509	0.4540
		75% perc. correlation	0.6185	0.4166
		90% perc. correlation	0.5300	0.3683
	10% perc. distance-to-default	Mean correlation	0.4914	0.3428
		75% perc. correlation	0.4630	0.3211
		90% perc. correlation	0.3757	0.2565

explaining the observed peaks whereas the AD ratios level is more relevant in other time.

6 Conclusion

The first question addressed by this paper is: How have banks' common exposures to shocks changed over the last decade in response to the changes in the environment in which the international banking sector operates? To answer this, we estimate the correlations between large international banks' asset-to-debt (AD) ratios over 1993-2006 with the flexible M-GARCH approach developed by Ledoit, Santa-Clara, and Wolf (2003). We find a decreasing trend until 2000, followed by an increasing trend. This suggests that, during the nineties, banks (or at least some of them) took advantage of the new technologies and markets available to them to pursue their own business strategies and to differentiate themselves from other banks, thus reducing their areas of common exposure. Since 2000, however, banks' areas of common exposure to shocks have increased rapidly, which could indicate that they are adopting increasingly similar strategies and moving into increasingly similar markets. This finding is also robust for different sub-groups of the sample.

The paper's second question concerns the impact of these trends on systemic risk in the banking sector. From a theoretical point of view, ongoing financial market integration and increasing cross-border activities may have both favourable and adverse effects on the stability of the banking system. To explore this question empirically, we construct a systemic risk index based on Lehar (2005), for which systemic risk is defined as the probability of a joint failure of a critical number of banks. As opposed to the correlation analysis, no clear trend emerges. Instead, we observe two peaks, one at the end of 1998 (LTCM and Russian crisis) and the other in 2002-2003 (stock market downturn), with the latter mainly doing damage to European banks.

The different patterns observed for banks' common exposure and for systemic risk contradict the widespread view that systemic risk increases with banks' co-movement. Our results confirm that the correlation between AD ratios is not a reliable measure for systemic risk. Instead, we find that the distance-to-default is the main driver of the systemic risk index. Once this distance-to-default is taken into account, however, correlation is positively associated with systemic risk.

These findings have two direct consequences for supervisory authorities: First, they show that systemic risk cannot be viewed as a pure correlation phenomenon. Instead, they highlight the danger of high and volatile leverages. According to our results, the main driver of systemic risk is the size of the risks taken by each bank individually (reflected by their distance-to-default) and not their common exposure to shocks (i.e. AD ratio correlation). Thus, supervisors concerned by systemic stability should first concentrate on making sure that banks are not taking disproportionate risks before trying to reduce inter-linkages or enforcing diversification in the banking sector. Second, from the monitoring point of view, co-movements between banks appear to be a spurious measure of systemic risk. Taken individually, this measure gives, in the best case, a weak indication about systemic risk and, in the worst case, may even point in the wrong direction. To be useful and unambiguous about the evolution of systemic risk, co-movement must be interpreted in combination with distance-to-default.

A Reduced form asset and debt dynamics

Developping each equation of the system

$$\begin{bmatrix} d\mathbf{A}_t \\ d\mathbf{D}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A}_t & 0 \\ 0 & \mathbf{D}_t \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_D \end{bmatrix} dt + \begin{bmatrix} \mathbf{A}_t & 0 \\ 0 & \mathbf{D}_t \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}_{AA} & \boldsymbol{\Upsilon}_{AD} \\ \boldsymbol{\Upsilon}_{DA} & \boldsymbol{\Upsilon}_{DD} \end{bmatrix} \begin{bmatrix} d\mathbf{w}_A \\ d\mathbf{w}_D \end{bmatrix}$$

yields

$$dA_t^i = A_t^i \mu_A^i dt + A_t^i \left(\sum_{j=1}^{n} \upsilon_{AA}^{ij} dw_A^j + \sum_{j=1}^{n} \upsilon_{AD}^{ij} dw_D^j \right) = A_t^i \mu_A^i dt + A_t^i \sigma_{A^i} dw_{A^i}$$

$$dD_t^i = D_t^i \mu_D^i dt + D_t^i \left(\sum_{j=1}^{n} \upsilon_{DA}^{ij} dw_A^j + \sum_{j=1}^{n} \upsilon_{DD}^{ij} dw_D^j \right) = D_t^i \mu_D^i dt + D_t^i \sigma_{D^i} dw_{D^i}$$

which are both Itô processes where

$$\sigma_{A^i} = \left(\sum_{j=1}^{n} \left(\upsilon_{AA}^{ij} \right)^2 + \sum_{j=1}^{n} \left(\upsilon_{AD}^{ij} \right)^2 \right)^{1/2}$$

and

$$\sigma_{D^i} = \left(\sum_{j=1}^{n} \left(\upsilon_{DA}^{ij} \right)^2 + \sum_{j=1}^{n} \left(\upsilon_{DD}^{ij} \right)^2 \right)^{1/2}$$

Using Itô's formula, we get that

$$d \ln A_t^i = \left(\mu_A^i - \frac{1}{2} \sigma_{A^i}^2 \right) dt + \sigma_{A^i} dw_{A^i}$$

$$d \ln D_t^i = \left(\mu_D^i - \frac{1}{2} \sigma_{D^i}^2 \right) dt + \sigma_{D^i} dw_{D^i}$$

To get the dynamic of dz_t^i, note that $z_t^i = \ln A_t^i - \ln D_t^i$, which implies that $dz_t^i = d \ln A_t^i - d \ln D_t^i$. Using the two previous equations in this expression gives

$$dz_t^i = \mu_z^i dt + \sigma_{z^i} dw_{z^i}$$

where

$$\mu_z^i = \mu_A^i - \mu_D^i - \frac{1}{2} \left(\sigma_{A^i}^2 - \sigma_{D^i}^2 \right)$$

$$\sigma_{z^i} = \left(\sum_{j=1}^{n} \left(\upsilon_{AA}^{ij} \right)^2 + \sum_{j=1}^{n} \left(\upsilon_{AD}^{ij} \right)^2 + \sum_{j=1}^{n} \left(\upsilon_{DA}^{ij} \right)^2 + \sum_{j=1}^{n} \left(\upsilon_{DD}^{ij} \right)^2 \right)^{1/2}$$

We can see that the dynamic of the AD ratio sums up the instantaneous growth rate of the bank's asset and debts as well as all interactions with shocks to its own and other banks' assets and debts.

Regrouping this equation for all the banks in one single system yields

$$d\mathbf{z}_t = \boldsymbol{\mu} dt + \boldsymbol{\Upsilon} d\mathbf{w}$$

B Simulation algorithm

To compute the systemic crisis index $I_t(\theta)$, we start the simulation at time t with the vector of AD ratio $\mathbf{z}_t$ and the estimated covariance matrix $\mathbf{H}_t$. The simulation uses the following algorithm:

(1) Generate a vector $\mathbf{e}_t$ containing N independent standard white noises. $\mathbf{e}_t$ simulates the independent shocks to the AD ratios.
(2) Generate the vector $\mathbf{u}_t$ from the vector $\mathbf{e}_t$ with a Choleski decomposition of the variance-covariance matrix $\mathbf{H}_t$. $\mathbf{u}_t$ is the total effect of the different individual shocks $\mathbf{e}_t$ on each AD ratio.
(3) Generate the new vector $\mathbf{z}_{t+1} = \mathbf{z}_t + \Delta\mathbf{z}_{t+1}$ with Equation (**??**).
(4) For each bank, check if $z_{t+1}^i < 0$ (insolvency condition). If bank i becomes insolvent, set $b_t^i = 1$.
(5) Check if there is a systemic crisis, i.e. if $\sum_{i=1}^N \theta_t^i b_t^i > \alpha$. Stop the algorithm if there is a systemic crisis.
(6) If there is no systemic crisis, compute the new variance-covariance matrix $\mathbf{H}_{t+1}$ with Equation (8), the vector $\mathbf{u}_t$ generated in Step 2 and the estimated coefficient matrices $\tilde{\mathbf{C}}$, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$.
(7) Repeat Steps 1 to 6 k times.

For each period, repeat this algorithm M times. The probability of a systemic crisis is then estimated by the number of times that the algorithm has stopped over M.

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Curriculum Vitae

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